Sequences

A **sequence** is an ordered collection of objects.

Example: A sequence of musical notes

Example: You may have seen a pattern recognition "IQ" puzzle similar to this:

△ ■ ○ ▲ □ ● △ _?__

In calculus, a **sequence** can be thought of as a list of numbers indexed by the natural numbers 1,2,3,4, … :

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

The number a_1 is called the first term, a_2 is the second term, and in general a_n is the n th term. The sequence $\{a_1,\ a_2,\ a_3,\ \quad\}$ is also denoted by

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}$$

Note that n doesn't have to start at 1.

Example: 1, 3, 5, 7, 9, … is a sequence of odd natural numbers.

A sequence can be finite or infinite, but for this class we're mostly interested in infinite sequences and what happens as you look further down the sequence.

Some sequences can be defined by giving an **explicit formula** for the n th term. For example, the **Geometric Sequence** (See homework Sec 11.1 p69):

$$a_n = a_1 r^{n-1}.$$

Each subsequent number is determined by multiplying the previous term by a fixed, nonzero number.

Example:
Write the first four terms of the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2n^2 - 3n + 1$.

Example: Find a formula for the general term a_n for the sequence of odd natural numbers.

Example:
Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5},\ -\frac{4}{25},\ \frac{5}{125},\ -\frac{6}{625},\ \frac{7}{3125},\ \quad\right\}$$

assuming that the pattern of the first few terms continues.

Example: 1,1,2,3,5,8,13, … the **Fibonacci Sequence** can be defined by:

$$a_1 = 1,\ a_2 = 1 \text{ and } a_{n+2} = a_n + a_{n+1} \text{ for all } n \geq 1.$$

Each subsequent number is the sum of the previous two.

Consider the sequence $a_n = \frac{n}{n+1}$. (Graph with Desmos)

It appears that the terms of the sequence $a_n = \frac{n}{n+1}$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large.

Definition

A sequence $\{a_n\}$ has the limit L and we write

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if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Example:

Determine whether the sequence $\{a_n\}$ where $a_n = \dfrac{n}{\sqrt{10+n}}$ is convergent or divergent.

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then __________________________ .

Example:

Determine whether the sequence $\{a_n\}$ where $a_n = \dfrac{\ln n}{n}$ is convergent or divergent.

Theorem

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

- $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.
- $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n$.
- $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$.
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ provided that $\lim\limits_{n\to\infty} b_n \neq 0$.

Theorem

If $a_n \le b_n \le c_n$ for $n \ge N$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Example:

Determine whether the sequence $\{a_n\}$ where $a_n = \dfrac{(-1)^n}{n}$ is convergent or divergent.

Theorem

If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Example:

Determine whether the sequence $\{a_n\}$ where $a_n = \sin\left(\dfrac{\pi}{n}\right)$ is convergent or divergent.