1. Question: Let $a_k = \dfrac{5k^2 - 9}{k^2 - 4}$ for $k = 3, 4, 5, \ldots$. The sequence $\{a_k\}_{k=3}^{\infty}$ converges to $L = \dfrac{5}{1}$. For each $\epsilon > 0$, find a positive number N such that $|a_k - 5| < \epsilon$ whenever $k > N$.

   Example solution: I have written two possible final answers in boxes (on page 2 and page 4) that you should imitate (note their short length). Before you can arrive to a final answer, you need to do some scratch work on a separate piece of paper. I have written some suggestions for how to write your scratch work.

   I described two options. Option 1 may be easier to follow at first because you would be doing step-by-step algebra. After you are comfortable with option 1, you can try option 2 - it is also doing algebra, but there is more room for creativity.

   Suggested scratch work (this is scratch work, so *do not submit this*):

   For all $k = 3, 4, 5, \ldots$, we have

$$\begin{aligned} |a_k - L| &= \left| \frac{5k^2 - 11}{k^2 - 4} - 5 \right| \\ &= \left| \frac{5k^2 - 11 - 5(k^2 - 4)}{k^2 - 4} \right| \\ &= \left| \frac{-11 + 20}{k^2 - 4} \right| \\ &= \left| \frac{9}{k^2 - 4} \right| \\ &= \frac{9}{k^2 - 4}. \end{aligned}$$

   From this point, I suggest you do one of two things (these are not the only options).

   Option 1:

   Think, I want to find k that is sufficiently large so that I get:

$$\begin{aligned} -\epsilon < a_k - L < \epsilon &\Leftrightarrow \\ -\epsilon < \frac{9}{k^2 - 4} < \epsilon &\Leftrightarrow \\ \frac{9}{k^2 - 4} < \epsilon &\Leftrightarrow \\ \frac{1}{k^2 - 4} < \frac{1}{9}\epsilon &\Leftrightarrow \\ 9\frac{1}{\epsilon} < k^2 - 4 & \text{ because both sides are positive} \Leftrightarrow \\ \frac{9}{\epsilon} + 4 < k^2 &\Leftrightarrow \\ \sqrt{\frac{9}{\epsilon} + 4} < k&. \end{aligned}$$

From the above computation, I can conclude that k would be large enough if $\sqrt{\frac{9}{\epsilon}+4} < k$. So I decide to choose my N to be $\sqrt{\frac{9}{\epsilon}+4}$ or larger. An example of a larger number would be $\sqrt{\frac{9}{\epsilon}+5}$.

(**STARTING MY FINAL ANSWER (to submit) if I like option 1 better**)

I choose $N = \sqrt{\frac{9}{\epsilon}+4}$ or $N = 3$ (whichever is bigger). Then, as long as $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{5k^2-11}{k^2-4} - 5\right| \\
&= \left|\frac{5k^2-11-5(k^2-4)}{k^2-4}\right| \\
&= \left|\frac{-11+20}{k^2-4}\right| \\
&= \left|\frac{9}{k^2-4}\right| \\
&= \frac{9}{k^2-4} \\
&< \frac{9}{N^2-4} \text{ because } k > N \\
&= \frac{9}{(\frac{9}{\epsilon}+4)-4} \text{ because I chose my } N \text{ to be } N = \sqrt{\frac{9}{\epsilon}+4} \\
&= \epsilon.
\end{aligned}$$

(**THE END OF MY FINAL ANSWER for option 1**)

Option 2: I first want to find a very simple-looking sequence which is bigger than $|a_k - L|$ for $k > C$, where C is some positive integer.

I want my simple-looking sequence to be

$$\frac{\text{some positive number A}}{k^2} \quad \text{or} \quad \frac{\text{some positive number B}}{k}.$$

You can guess some values or go through the following step-by-step process which guarantees you to find values that would work.

I want to find A and C so that the following is true:

$$\begin{aligned}
|a_k - L| &< \frac{A}{k^2} \text{ for } k > C \iff \\
\left|\frac{9}{k^2-4}\right| &< \frac{A}{k^2} \text{ for } k \geq C \iff \\
\frac{9}{k^2-4} &< \frac{A}{k^2} \text{ for } k \geq C \iff \\
\frac{9}{A}\, k^2 &< (k^2-4) \text{ for } k \geq C
\end{aligned}$$

In order for above to be true, I see that $A(k^2-4)$ must be positive, so k^2 must be bigger than 4. Hence, I should choose C to be larger than $\sqrt{4}$. I choose $C := 3$ (or I could have chosen $C := 100$).

Now I can choose my A. I need:

$$\begin{aligned}
\frac{9}{A}\, k^2 &< (k^2-4) \text{ for } k \geq 3 && \iff \\
4 &< k^2 - \frac{9}{A}\, k^2 \text{ for } k \geq 3 && \iff \\
4 &< \left(1 - \frac{9}{A}\right) k^2 \text{ for } k \geq 3 && \iff
\end{aligned}$$

I see that the last equation will be true for all $k \geq 3$ if

$$4 < \left(1 - \frac{9}{A}\right) 3^2 \iff \tag{1}$$

$$4 < 9 - \frac{81}{A} \iff \tag{2}$$

$$\frac{81}{A} < 9 - 4 \iff \tag{3}$$

$$\frac{81}{5} < A \tag{4}$$

is true. So I want to choose my A so that the inequality in line (4) is true. The inequality in line (4) is true as long as A is bigger than 81/7. A convenient number larger than 81/5 is $85/5 = 17$. I choose $A = 17$.

All the work above get us:

$$\begin{aligned}
|a_k - L| &= \left|\frac{9}{k^2-4}\right| \\
&< \frac{17}{k^2} \text{ for } k \geq 3
\end{aligned}$$

Think, I want to find k that is sufficiently large so that I get:

$$\frac{17}{k^2} < \epsilon \Leftrightarrow$$
$$\frac{1}{k^2} < \frac{1}{17}\,\epsilon$$
$$\frac{17}{\epsilon} < k^2 \text{ because both sides are positive} \Leftrightarrow$$
$$\sqrt{\frac{17}{\epsilon}} < k.$$

From the above computation, I can conclude that k would be large enough if $\sqrt{\frac{17}{\epsilon}} < k$. So I decide to choose my N to be $\sqrt{\frac{17}{\epsilon}}$. I could have chosen a larger one, say, $\frac{5}{\sqrt{\epsilon}}$.

(STARTING MY FINAL ANSWER (to submit) if I like option 2 better)
I choose $N = \sqrt{17/\epsilon}$ or $N = 3$ (whichever is bigger). Then, as long as $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{5k^2-11}{k^2-4} - 5\right| \\
&= \left|\frac{5k^2-11-5(k^2-4)}{k^2-4}\right| \\
&= \left|\frac{-11+20}{k^2-4}\right| \\
&= \left|\frac{9}{k^2-4}\right| \\
&= \frac{9}{k^2-4} \\
&< \frac{17}{k^2} \text{ because } k > N \geq 3 \\
&< \frac{17}{N^2} \text{ because } k > N \\
&= \frac{17}{(17/\epsilon)} \text{ because I set my } N = \sqrt{17/\epsilon} \\
&= \epsilon.
\end{aligned}$$

(THE END OF MY FINAL ANSWER for option 2)