Taylor and Maclaurin Series

Suppose for $|x-a|<R$, we have

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+\quad\dots,$$

then $f(a)=c_0$. We can differentiate both sides with respect to x to get

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+\quad\dots,$$

then $f'(a)=c_1$. Again, we have

$$f''(x)=2c_2+2\times3c_3(x-a)+3\times4c_4(x-a)^2+\quad\dots,$$

then $f''(a)=2c_2$. Apply the procedure one more time to obtain

$$f'''(x)=2\times3c_3+2\times3\times4c_4(x-a)+\quad\dots,$$

then $f'''(a)=2\times3c_3$. By now you can see the pattern. If we continue to differentiate and substitute $x=a$, we obtain

$$f^{(n)}(a)=n!\;c_n.$$

That is

$$c_n=\frac{f^{(n)}(a)}{n!}.$$

<u>Example</u>:

If $f(x)=\sum_{n=0}^{\infty}c_n(x-5)^n$ for all x, write a formula for c_8. Answer: follow Theorem 5 (pg 760).

Theorem (Sec 11.10 page 760)

If f has a power series representation at $x = a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for} \qquad |x-a| < R,$$

then its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \end{aligned}$$

The series is called the **Taylor series** of the function f at $x = a$. For the special case when $a = 0$, the Taylor series becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \quad \dots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

Example (check solution of Example 1, pg 760):
Find the **Maclaurin series** of the function $f(x) = e^x$ and its interval of convergence.

Recall the proof that *e* is irrational (from Sec 11.5, Alternating Series Estimate Theorem):
We needed the fact that *1/e* is equal to the series ____________________ ?

Consider the partial sums

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Notice that T_n is a polynomial of degree n called the nth-degree **Taylor polynomial** of f at $x = a$.

<u>Example</u>:

Consider the function $f(x) = \ln x$.

a. Find the **<u>Taylor polynomial of order 2</u>** centered at $x = 1$ for $f(x) = \ln x$.

b. Estimate the value of $\ln 1.05$.