L'Hopital's Rule

The Indeterminate Forms Family (from Sec 4.4 p305)

$$\frac{0}{0},\ \frac{\infty}{\infty} \text{ and } 0\cdot\infty \text{ are } \textbf{indeterminate forms}.$$

L'Hopital's Rule for $\frac{0}{0}$

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right side exists.
The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.

Caution

- L'Hopital's Rule is **NOT** Quotient Rule.
- You must get the indeterminate form to apply L'Hopital's Rule.

Example:

Evaluate $\lim_{x\to -1}\frac{x^3-4x^2-11x-6}{x^3+8x^2+13x+6}$.

[Solution]

$$\lim_{x\to -1}\frac{x^3-4x^2-11x-6}{x^3+8x^2+13x+6}, \text{ an Indeterminate Form } \frac{0}{0}$$
$$\overset{L}{=}\lim_{x\to -1}\frac{3x^2-8x-11}{3x^2+16x+13}, \text{ an Indeterminate Form } \frac{0}{0}$$
$$\overset{L}{=}\lim_{x\to -1}\frac{6x-8}{6x+16}$$
$$=\frac{-6-8}{-6+16}$$
$$=-\frac{7}{5}$$

L'Hopital's Rule for $\frac{\infty}{\infty}$

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Provided the limit on the right side exists.
The rule also applies if $x \to a$ is replaced by $x \to \pm\infty$, $x \to a^+$ or $x \to a^-$.

Example:

Evaluate $\lim_{x \to \infty} \frac{16x^2 - 8x - 6}{18x^2 - 6x + 8}$.

[Solution]

$\lim_{x \to \infty} \frac{16x^2 - 8x - 6}{18x^2 - 6x + 8}$, an Indeterminate Form $\frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x \to \infty} \frac{32x - 8}{36x - 6}$, an Indeterminate Form $\frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x \to \infty} \frac{32}{36}$

$= \frac{8}{9}$

L'Hopital's Rule for $0 \cdot \infty$

If we are asked to evaluate

$$\lim_{x \to a} f(x) g(x),$$

where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$.

WE CANNOT APPLY L'HOPITAL'S RULE DIRECTLY.

We need to use algebra to get either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\lim_{x \to a} f(x) g(x) = \lim_{x \to a} \frac{f(x) g(x)}{1}$$

$$= \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}}, \text{ an Indeterminate Form } \frac{0}{0}$$

OR

$$= \lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}}, \text{ an Indeterminate Form } \frac{\infty}{\infty}$$

Example:

Evaluate $\lim_{x \to \infty} \left[x \sin\left(\frac{16}{x}\right) \right]$.

[Solution]

$$\lim_{x \to \infty} \left[x \sin\left(\frac{16}{x}\right) \right]$$

$$= \lim_{x \to \infty} \left[\frac{\sin\left(\frac{16}{x}\right)}{\frac{1}{x}} \right], \text{ an Indeterminate Form } \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to \infty} \left[\frac{\cos\left(\frac{16}{x}\right) \cdot \left(\frac{-16}{x^2}\right)}{\frac{-1}{x^2}} \right]$$

$$= \lim_{x \to \infty} \left[16 \cos\left(\frac{16}{x}\right) \right]$$

$$= 16$$

The Indeterminate Forms Family (in Sec 4.4 page 310)

The **indeterminate forms** 1^∞, 0^0 and ∞^0 all arise in limits of the form

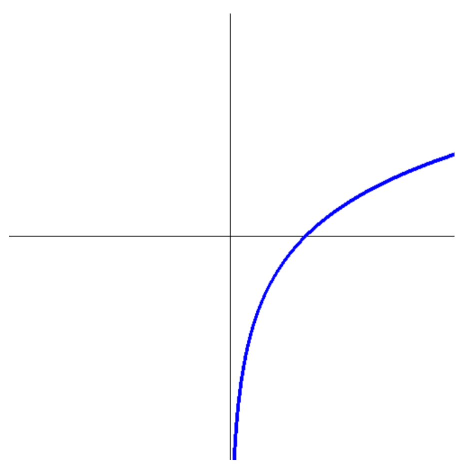
$$\lim_{x\to a} f(x)^{g(x)}.$$

Procedure

Suppose $\lim_{x\to a} f(x)^{g(x)}$ has the indeterminate form 1^∞, 0^0 or ∞^0.

- Let $y = f(x)^{g(x)}$. Then $\ln y = g(x)\ln f(x)$.
- Evaluate $\lim_{x\to a}\ln y$. This limit can be put in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, both of which are handled by L'Hôpital's Rule.
- Then $\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = e^{\lim_{x\to a}\ln y}$.

Useful Information about Natural Logarithmic Function



- $\lim_{x\to 0^+}\ln x = -\infty$. $\lim_{x\to 1}\ln x = 0$. $\lim_{x\to \infty}\ln x = \infty$.
- $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}\cdot f'(x)$.

<u>Example</u>:

Evaluate $\lim_{x\to 0}(1+4x)^{\frac{3}{x}}$.

[Solution]

$\lim_{x\to 0}(1+4x)^{\frac{3}{x}}$ is an Indeterminate Form 1^{∞}.

Let $y=(1+4x)^{\frac{3}{x}}$,

then
$$\begin{aligned}\ln y &= \ln(1+4x)^{\frac{3}{x}}\\ &= \frac{3}{x}\cdot\ln(1+4x)\\ &= \frac{3\ln(1+4x)}{x}\end{aligned}$$

$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{3\ln(1+4x)}{x}$, an Indeterminate Form $\frac{0}{0}$, so L'Hôpital's Rule applies

$$\begin{aligned}&\overset{L}{=}\lim_{x\to 0}\frac{3\cdot\frac{1}{1+4x}\cdot 4}{1}\\ &=12.\end{aligned}$$

Therefore,
$$\begin{aligned}\lim_{x\to 0}(1+4x)^{\frac{3}{x}} &= \lim_{x\to 0}y\\ &= \lim_{x\to 0}e^{\ln y}\\ &= e^{\lim_{x\to 0}\ln y}\\ &= e^{12}.\end{aligned}$$

Example:

Evaluate $\lim_{x\to 0^+}(\sin x)^{\tan x}$.

[Solution]

$\lim_{x\to 0^+}(\sin x)^{\tan x}$ is an Indeterminate Form 0^0.

Let $y=(\sin x)^{\tan x}$,

then $\ln y = \ln(\sin x)^{\tan x}$

$= \tan x \cdot \ln(\sin x)$

Suppose that we write $\tan x \cdot \ln(\sin x)$ as $\dfrac{\sin x \cdot \ln(\sin x)}{\cos x}$.

$\lim_{x\to 0^+}\dfrac{\sin x \cdot \ln(\sin x)}{\cos x}$ is an Indeterminate Form $0\cdot(-\infty)$,

we cannot apply L'Hôpital's Rule.

Therefore, write $\tan x \cdot \ln(\sin x)$ as $\dfrac{\ln(\sin x)}{\cot x}$.

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\cot x}$$, an Indeterminate Form $\dfrac{-\infty}{\infty}$, so L'Hôpital's Rule applies

$$\overset{L}{=} \lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cdot\cos x}{-\csc^2 x}$$

$$= \lim_{x\to 0^+}(-\sin x\cos x)$$

$$= 0.$$

Therefore, $\lim_{x\to 0^+}(\sin x)^{\tan x} = \lim_{x\to 0^+}\ln y$

$$= \lim_{x\to 0^+} e^{\ln y}$$

$$= e^{\lim_{x\to 0^+}\ln y}$$

$$= e^0$$

$$= 1.$$