Name : ______________________________

1. Evaluate $\int \frac{1}{x\sqrt{4-x^2}}\,dx$.

[Solution]

Let $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $dx = 2\cos\theta\, d\theta$.

Thus
$$\begin{aligned}\int \frac{1}{x\sqrt{4-x^2}}\,dx &= \int \frac{1}{2\sin\theta\sqrt{4-4\sin^2\theta}}\,2\cos\theta\, d\theta \\ &= \frac{1}{2}\int \frac{1}{\sin\theta}\,d\theta \\ &= \frac{1}{2}\int \csc\theta\, d\theta \\ &= -\frac{1}{2}\ln\left|\csc\theta+\cot\theta\right| + C \\ &= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C\end{aligned}$$

2. Evaluate $\int \frac{1}{\sqrt{x^2+16}}\,dx$.

[Solution]

Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = 4\sec^2\theta\, d\theta$.

Thus
$$\begin{aligned}\int \frac{1}{\sqrt{x^2+16}}\,dx &= \int \frac{1}{\sqrt{16\tan^2\theta+16}}\,4\sec^2\theta\, d\theta \\ &= \int \sec\theta\, d\theta \\ &= \ln\left|\sec\theta+\tan\theta\right| + C \\ &= \ln\left|\frac{\sqrt{x^2+16}+x}{4}\right| + C\end{aligned}$$

3. Evaluate $\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,dx$.

[Solution]

Let $x = \sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, then $dx = \sec\theta\tan\theta\,d\theta$.

When $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$ and when $x = 2$, $\theta = \frac{\pi}{3}$.

Thus
$$\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^3\theta\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta}\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\cos 2\theta}{2}\,d\theta$$
$$= \left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \left(\frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3}\right) - \left(\frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2}\right)$$
$$= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$

4. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$.

[Solution]

Let $u=\sin x$, then $du=\cos x\,dx$.

When $x=0$, $u=0$ and when $x=\frac{\pi}{2}$, $u=1$.

Thus $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx=\int_0^1 \frac{1}{\sqrt{1+u^2}}\,du$

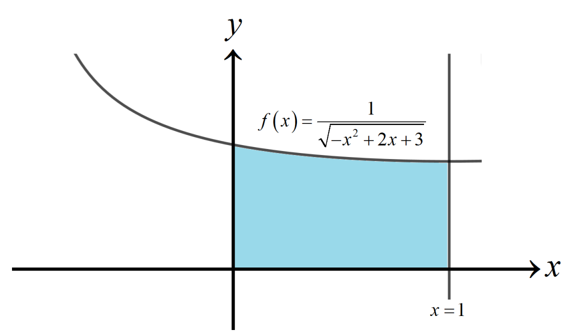
Let $u=\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then $du=\sec^2\theta\,d\theta$.

When $u=0$, $\theta=0$ and when $u=1$, $\theta=\frac{\pi}{4}$.

$$\begin{aligned}
\text{Thus } \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx &= \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta \\
&= \left(\ln\left|\sec\theta+\tan\theta\right|\right)_0^{\frac{\pi}{4}} \\
&= \ln\left(\sqrt{2}+1\right)
\end{aligned}$$

5. Let R be the region bounded by the function $f(x)=\frac{1}{\sqrt{-x^2+2x+3}}$ and x-axis on the interval $[0,1]$.

Find the area of the region R.



[Solution] 1

$$A=\int_0^1 \frac{1}{\sqrt{-x^2+2x+3}}\,dx$$

$$\begin{aligned}
-x^2+2x+3 &= -\left(x^2-2x\right)+3 \\
&= -\left(x^2-2x+1\right)+1+3 \\
&= -(x-1)^2+4
\end{aligned}$$

Thus $A = \int_0^1 \frac{1}{\sqrt{-x^2+2x+3}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}\,dx$

Let $x-1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $dx = 2\cos\theta\,d\theta$.

When $x=0$, $\theta = -\frac{\pi}{6}$ and when $x=1$, $\theta = 0$.

Thus $A = \int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}\,dx$

$= \int_{-\frac{\pi}{6}}^{0} \frac{1}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\,d\theta$

$= \int_{-\frac{\pi}{6}}^{0} 1\,d\theta$

$= \frac{\pi}{6}$

[Solution] 2

$A = \int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}\,dx$

$= \int_0^1 \frac{1}{2\sqrt{1-\left(\frac{x-1}{2}\right)^2}}\,dx$

Let $u = \frac{x-1}{2}$, then $du = \frac{1}{2}\,dx$.

When $x=0$, $u = -\frac{1}{2}$ and when $x=1$, $u=0$.

Thus $A = \int_0^1 \frac{1}{2\sqrt{1-\left(\frac{x-1}{2}\right)^2}}\,dx$

$= \int_{-\frac{1}{2}}^{0} \frac{1}{\sqrt{1-u^2}}\,du$

$= \left(\sin^{-1} u\right)_{-\frac{1}{2}}^{0}$

$= \frac{\pi}{6}$