ANSWER KEY for homework due Week 6 Mon, Oct 2, 2017 Fall 17

Name : ________________________________

1. Evaluate $\int \ln\left(x+\sqrt{1+x^2}\right)\,dx$.

[Solution]

Let $u=\ln\left(x+\sqrt{1+x^2}\right)$ and $dv=dx$,

$$\text{then } du=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)dx$$
$$=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\,dx$$
$$=\frac{1}{\sqrt{1+x^2}}\,dx$$

and $v=x$.

Thus $\int \ln\left(x+\sqrt{1+x^2}\right)dx = x\ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x}{\sqrt{1+x^2}}\,dx$

For $\int\frac{x}{\sqrt{1+x^2}}\,dx$, let $w=1+x^2$, then $dw=2x\,dx$.

$$\text{Thus } \int\frac{x}{\sqrt{1+x^2}}\,dx=\int\frac{1}{2\sqrt{w}}\,dw$$
$$=\sqrt{w}+C$$
$$=\sqrt{1+x^2}+C$$

Therefore, $\int \ln\left(x+\sqrt{1+x^2}\right)dx = x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C$

2. Evaluate $\int x\tan^2 x\,dx$.

[Solution]

$$\int x\tan^2 x\,dx=\int x\left(\sec^2 x-1\right)dx$$
$$=\int x\sec^2 x\,dx-\int x\,dx$$

For $\int x\sec^2 x\,dx$, let $u=x$ and $dv=\sec^2 x\,dx$,

then $du=dx$ and $v=\tan x$.

$$\text{Thus } \int x\sec^2 x\,dx = x\tan x-\int \tan x\,dx$$
$$=x\tan x-\int\frac{\sin x}{\cos x}\,dx$$

For $\int \frac{\sin x}{\cos x}\,dx$, let $w = \cos x$, then $dw = -\sin x\,dx$.

Thus $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$$= -\int \frac{1}{w}\,dw$$
$$= -\ln|w| + C$$
$$= -\ln|\cos x| + C$$

Therefore, $\int x\tan^2 x\,dx = \int x\sec^2 x\,dx - \int x\,dx$

$$= \left(x\tan x - \int \tan x\,dx\right) - \frac{1}{2}x^2$$
$$= x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

3. Evaluate $\int \cos\sqrt{x}\,dx$

[Solution]

Let $w = \sqrt{x}$, then $dw = \frac{1}{2\sqrt{x}}\,dx$.

Thus $\int \cos\sqrt{x}\,dx = 2\int w\cos w\,dw$

For $\int w\cos w\,dw$, let $u = w$ and $dv = \cos w\,dw$,

then $du = dw$ and $v = \sin w$.

Thus $\int w\cos w\,dw = w\sin w - \int \sin w\,dw$

$$= w\sin w + \cos w + c$$

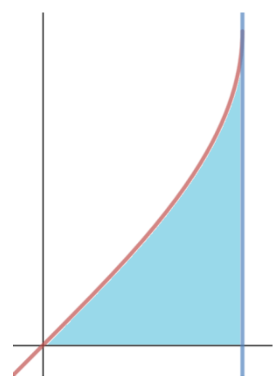
Therefore, $\int \cos\sqrt{x}\,dx = 2\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x} + C$

4. Evaluate $\int x^2 (\ln x)^2 \, dx$.
   [Solution]
   Let $u = (\ln x)^2$ and $dv = x^2 \, dx$,
   then $du = \dfrac{2\ln x}{x} \, dx$ and $v = \dfrac{1}{3}x^3$.
   Thus $\int x^2 (\ln x)^2 \, dx = \dfrac{1}{3}x^3 (\ln x)^2 - \dfrac{2}{3}\int x^2 \ln x \, dx$

   For $\int x^2 \ln x \, dx$, let $s = \ln x$ and $dt = x^2 \, dx$,
   then $ds = \dfrac{1}{x} \, dx$ and $t = \dfrac{1}{3}x^3$.
   Thus $\int x^2 \ln x \, dx = \dfrac{1}{3}x^3 \ln x - \dfrac{1}{3}\int x^2 \, dx$

   Therefore,
$$\begin{aligned}\int x^2 (\ln x)^2 \, dx &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\left(\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx\right) \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C\end{aligned}$$

5. Consider the graph of the function $f(x) = \sin^{-1} x$. Let R be the region bounded by $y = f(x)$ and x-axis on the interval $[0,1]$.



   Evaluate the **area** of R.
   [Solution]
   $A = \int_0^1 \sin^{-1} x \, dx$

   Let $u = \sin^{-1} x$ and $dv = dx$,
   then $du = \dfrac{1}{\sqrt{1-x^2}} \, dx$ and $v = x$.
   Thus $\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \dfrac{x}{\sqrt{1-x^2}} \, dx$

For $\int \frac{x}{\sqrt{1-x^2}}\,dx$, let $w=1-x^2$, then $dw=-2x\,dx$.

Thus $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{2\sqrt{w}}\,dw$

$= -\sqrt{w}+C$

$= -\sqrt{1-x^2}+c$

Therefore, $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1} x + \sqrt{1-x^2} + C$

As a result, $A = \int_0^1 \sin^{-1} x\,dx$

$= \left(x\sin^{-1} x + \sqrt{1-x^2}\right)\Big|_0^1$

$= \sin^{-1} 1 - 1$

$= \frac{\pi}{2} - 1$

6. Evaluate $\int \cos(\ln x)\,dx$.

[Solution]

Let $u=\cos(\ln x)$ and $dv=dx$,

then $du = \frac{-\sin(\ln x)}{x}\,dx$ and $v=x$.

Thus $\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

For $\int \sin(\ln x)\,dx$, let $s=\sin(\ln x)$ and $dt=dx$,

then $ds = \frac{\cos(\ln x)}{x}\,dx$ and $t=x$.

Thus $\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$

Therefore, $\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

$= x\cos(\ln x) + \left[x\sin(\ln x) - \int \cos(\ln x)\,dx\right]$

As a result, $\int \cos(\ln x)\,dx = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C$