1. Evaluate $\int \frac{x-9}{x^2-3x-18}\,dx$.

[Solution] Since $x^2-3x-18=(x-6)(x+3)$,

we can let $\frac{x-9}{x^2-3x-18}=\frac{A}{x-6}+\frac{B}{x+3}$.

Multiply both sides by the denominator to get

$$x-9=A(x+3)+B(x-6)$$
$$=(A+B)x+(3A-6B)$$

Hence, $\begin{cases} A+B=1 \\ 3A-6B=-9 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{3} \\ B=\frac{4}{3} \end{cases}$

Thus, $\frac{x-9}{x^2-3x-18}=-\frac{1}{3}\cdot\frac{1}{x-6}+\frac{4}{3}\cdot\frac{1}{x+3}$.

Therefore, $\int \frac{x-9}{x^2-3x-18}\,dx=\int\left(-\frac{1}{3}\cdot\frac{1}{x-6}+\frac{4}{3}\cdot\frac{1}{x+3}\right)dx$

$$=-\frac{1}{3}\int\frac{1}{x-6}\,dx+\frac{4}{3}\int\frac{1}{x+3}\,dx$$

$$=-\frac{1}{3}\ln|x-6|+\frac{4}{3}\ln|x+3|+K \text{ where } K \text{ is a constant}$$

2. Evaluate $\int \frac{2}{x^3+x^2}\,dx$.

[Solution] Since $x^3+x^2=x^2(x+1)$,

we can let $\frac{2}{x^3+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$.

Multiply both sides by the denominator to get $2=Ax(x+1)+B(x+1)+Cx^2$

$$=(A+C)x^2+(A+B)x+B$$

Hence, $\begin{cases} A+C=0 \\ A+B=0 \\ B=2 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=2 \\ C=2 \end{cases}$

Thus, $\frac{2}{x^3+x^2}=-2\cdot\frac{1}{x}+2\cdot\frac{1}{x^2}+2\cdot\frac{1}{x+1}$.

Therefore, $\int \frac{2}{x^3+x^2}\,dx=\int\left(-2\cdot\frac{1}{x}+2\cdot\frac{1}{x^2}+2\cdot\frac{1}{x+1}\right)dx$

$$=-2\int\frac{1}{x}\,dx+2\int\frac{1}{x^2}\,dx+2\int\frac{1}{x+1}\,dx$$

$$=-2\ln|x|-\frac{2}{x}+2\ln|x+1|+K \text{ where } K \text{ is a constant}$$

3. Evaluate $\int \frac{1}{x^4+3x^2+2}\,dx$.

[Solution]

Since $x^4+3x^2+2=(x^2+1)(x^2+2)$,

we can let $\frac{1}{x^4+3x^2+2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}$.

Multiply both sides by the denominator to get

$$1=(Ax+B)(x^2+2)+(Cx+D)(x^2+1)$$
$$=(A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$$

Hence, $\begin{cases} A+C=0 \\ B+D=0 \\ 2A+C=0 \\ 2B+D=1 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=0 \\ D=-1 \end{cases}$

Thus, $\frac{1}{x^4+3x^2+2}=\frac{1}{x^2+1}-\frac{1}{x^2+2}$.

Therefore, $\int \frac{1}{x^4+3x^2+2}\,dx=\int\left(\frac{1}{x^2+1}-\frac{1}{x^2+2}\right)dx$

$$=\int\frac{1}{x^2+1}\,dx-\int\frac{1}{x^2+2}\,dx$$

$$=\arctan(x)-\frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+K \text{ where } K \text{ is a constant}$$

4. Evaluate $\int \frac{3x^2+4x-6}{x^2-3x+2}\,dx$.

[Solution]

Note that $\frac{3x^2+4x-6}{x^2-3x+2} = 3 + \frac{13x-12}{x^2-3x+2}$

Since $x^2-3x+2=(x-1)(x-2)$,

we can let $\frac{13x-12}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$.

Multiply both sides by the denominator to get

$$\begin{aligned} 13x-12 &= A(x-2)+B(x-1) \\ &= (A+B)x+(-2A-B) \end{aligned}$$

Hence, $\begin{cases} A+B=13 \\ -2A-B=-12 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=14 \end{cases}$

Thus, $\frac{13x-12}{x^2-3x+2} = -\frac{1}{x-1} + 14\cdot\frac{1}{x-2}$.

Therefore,

$$\begin{aligned} \int \frac{3x^2+4x-6}{x^2-3x+2}\,dx &= \int\left(3+\frac{13x-12}{x^2-3x+2}\right)dx \\ &= \int\left(3-\frac{1}{x-1}+14\cdot\frac{1}{x-2}\right)dx \\ &= \int 3\,dx - \int\frac{1}{x-1}\,dx + 14\int\frac{1}{x-2}\,dx \\ &= 3x - \ln|x-1| + 14\ln|x-2| + K \text{ where } K \text{ is a constant} \end{aligned}$$

5. Evaluate $\int \frac{\sec\theta}{1+\sin\theta}\, d\theta$.

[Solution] Since
$$\frac{\sec\theta}{1+\sin\theta} = \frac{1}{(1+\sin\theta)\cos\theta} = \frac{\cos\theta}{(1+\sin\theta)\cos^2\theta} = \frac{\cos\theta}{(1+\sin\theta)(1-\sin^2\theta)} = \frac{\cos\theta}{(1+\sin\theta)^2(1-\sin\theta)}$$

we can let $u=\sin\theta$ and $du=\cos\theta\, d\theta$.

Thus
$$\int \frac{\sec\theta}{1+\sin\theta}\, d\theta = \int \frac{\cos\theta}{(1+\sin\theta)^2(1-\sin\theta)}\, d\theta = \int \frac{1}{(1+u)^2(1-u)}\, du$$

Furthermore, let $\frac{1}{(1+u)^2(1-u)} = \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{1-u}$

Multiply both sides by the denominator
$$1 = A(1+u)(1-u) + B(1-u) + C(1+u)^2 = (-A+C)u^2 + (-B+2C)u + (A+B+C)$$

Hence,
$$\begin{cases} -A+C=0 \\ -B+2C=0 \\ A+B+C=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=\frac{1}{2} \\ C=\frac{1}{4} \end{cases}$$

Thus, $\frac{1}{(1+u)^2(1-u)} = \frac{1}{4}\cdot\frac{1}{1+u} + \frac{1}{2}\cdot\frac{1}{(1+u)^2} + \frac{1}{4}\cdot\frac{1}{1-u}$.

Therefore,
$$\begin{aligned}\int \frac{\sec\theta}{1+\sin\theta}\, d\theta &= \int \frac{1}{(1+u)^2(1-u)}\, du \\ &= \int \left[\frac{1}{4}\cdot\frac{1}{1+u} + \frac{1}{2}\cdot\frac{1}{(1+u)^2} + \frac{1}{4}\cdot\frac{1}{1-u}\right] du \\ &= \frac{1}{4}\int \frac{1}{1+u}\, du + \frac{1}{2}\int \frac{1}{(1+u)^2}\, du + \frac{1}{4}\int \frac{1}{1-u}\, du \\ &= \frac{1}{4}\ln|1+u| - \frac{1}{2}\cdot\frac{1}{1+u} - \frac{1}{4}\ln|1-u| + K \\ &= \frac{1}{4}\ln|1+\sin\theta| - \frac{1}{2}\cdot\frac{1}{1+\sin\theta} - \frac{1}{4}\ln|1-\sin\theta| + K\end{aligned}$$
where K is a constant

6. Evaluate $\int \frac{x+1}{x^3+4x}\,dx$.

[Solution]

Since $x^3+4x = x\left(x^2+4\right)$,

we can let $\frac{x+1}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$.

Multiply both sides by the denominator to get

$$x+1 = A\left(x^2+4\right) + \left(Bx+C\right)x$$
$$= \left(A+B\right)x^2 + Cx + 4A$$

Hence, $\begin{cases} A+B=0 \\ C=1 \\ 4A=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=1 \end{cases}$

Thus, $\frac{x+1}{x^3+4x} = \frac{1}{4}\cdot\frac{1}{x} + \frac{-\frac{1}{4}x+1}{x^2+4}$

$$= \frac{1}{4}\cdot\frac{1}{x} - \frac{1}{4}\cdot\frac{x}{x^2+4} + \frac{1}{x^2+4}$$

Therefore, $\int \frac{x+1}{x^3+4x}\,dx = \int\left(\frac{1}{4}\cdot\frac{1}{x} - \frac{1}{4}\cdot\frac{x}{x^2+4} + \frac{1}{x^2+4}\right)dx$

$$= \frac{1}{4}\int\frac{1}{x}\,dx - \frac{1}{4}\int\frac{x}{x^2+4}\,dx + \int\frac{1}{x^2+4}\,dx$$

$$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln\left(x^2+4\right) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K$$ where K is a constant