Name : ________________________________

1. Evaluate $\int \frac{1}{\sqrt{1+16x^2}}\,dx$.

   [Solution]

   Let $x=\frac{1}{4}\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then $dx=\frac{1}{4}\sec^2\theta\,d\theta$.

   Thus
$$\int \frac{1}{\sqrt{1+16x^2}}\,dx = \int \frac{1}{\sqrt{1+\tan^2\theta}}\frac{1}{4}\sec^2\theta\,d\theta$$
$$=\frac{1}{4}\int \sec\theta\,d\theta$$
$$=\frac{1}{4}\ln\left|\sec\theta+\tan\theta\right|+C$$
$$=\frac{1}{4}\ln\left|\sqrt{1+16x^2}+4x\right|+C$$

2. Evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$.

   [Solution]

   Let $x=2\sin\theta$ where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, then $dx=2\cos\theta\,d\theta$.

   When $x=0$, $\theta=0$ and when $x=\sqrt{2}$, $\theta=\frac{\pi}{4}$.

   Thus
$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,d\theta$$
$$=\int_0^{\frac{\pi}{4}} 4\sin^2\theta\,d\theta$$
$$=4\int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\,d\theta$$
$$=\left(2\theta-\sin 2\theta\right)_0^{\frac{\pi}{4}}$$
$$=\frac{\pi}{2}-1$$

3. Evaluate $\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx$.

[Solution]

Let $x=\frac{1}{3}\sec\theta$ where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$, then $dx=\frac{1}{3}\sec\theta\tan\theta\,d\theta$.

Thus
$$\begin{aligned}\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx &= \int \frac{1}{\frac{1}{9}\sec^2\theta\sqrt{\sec^2\theta-1}}\frac{1}{3}\sec\theta\tan\theta\,d\theta\\ &=3\int\frac{1}{\sec\theta}\,d\theta\\ &=3\int\cos\theta\,d\theta\\ &=3\sin\theta+C\\ &=\frac{\sqrt{9x^2-1}}{x}+C\end{aligned}$$

4. Evaluate $\int\sqrt{5+4x-x^2}\,dx$.

[Solution]

$$\begin{aligned}5+4x-x^2&=-\left(x^2-4x\right)+5\\ &=-\left(x^2-4x+4\right)+4+5\\ &=-\left(x-2\right)^2+9\end{aligned}$$

Let $x-2=3\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, then $dx=3\cos\theta\,d\theta$.

Thus
$$\begin{aligned}\int\sqrt{5+4x-x^2}\,dx&=\int\sqrt{9-\left(x-2\right)^2}\,dx\\ &=\int 3\cos\theta\sqrt{9-9\sin^2\theta}\,d\theta\\ &=9\int\cos^2\theta\,d\theta\\ &=9\int\frac{1+\cos2\theta}{2}\,d\theta\\ &=\frac{9}{2}\theta+\frac{9}{4}\sin2\theta+C\\ &=\frac{9}{2}\theta+\frac{9}{2}\sin\theta\cos\theta+C\\ &=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{9-\left(x-2\right)^2}}{3}+C\\ &=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{x-2}{2}\sqrt{9-\left(x-2\right)^2}+C\end{aligned}$$