1. To derive the formula for **Integration by Parts** we used which of the following theorems?

   1) The Fundamental Theorem of Calculus.

   2) The Product Rule of Differentiation.

   3) The Chain Rule of Differentiation.

   4) The Mean Value Theorem

2. Evaluate $\int_0^{\frac{\pi}{2}} x\cos 2x\, dx$.

   [Solution]

   Let $u = x$ and $dv = \cos 2x\, dx$,

   then $du = dx$ and $v = \frac{1}{2}\sin 2x$.

   Thus $\int_0^{\frac{\pi}{2}} x\cos 2x\, dx = \frac{1}{2}\left(x\sin 2x\right)_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

   $$= \frac{1}{2}\left[\frac{\pi}{2}\sin\pi - 0\right] - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)_0^{\frac{\pi}{2}}$$

   $$= \frac{1}{4}\left(\cos\pi - \cos 0\right)$$

   $$= -\frac{1}{2}$$

3. Suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$ and f'' is continuous. Evaluate $\int_1^4 x f''(x)\, dx$.

[Solution]

Let $u=x$ and $dv=f''(x)\, dx$,

then $du=dx$ and $v=f'(x)$.

Thus $\int_1^4 x f''(x)\, dx = xf'(x)\Big|_1^4 - \int_1^4 f'(x)\, dx$

$$= \left[4f'(4)-f'(1)\right]-\left[f(4)-f(1)\right]$$
$$=2$$

4. Evaluate $\int \tan^{-1} x\, dx$.

[Solution]

Let $u=\tan^{-1} x$ and $dv=dx$,

then $du=\dfrac{1}{1+x^2}\, dx$ and $v=x$.

Thus $\int \tan^{-1} x\, dx = x\tan^{-1} x - \int \dfrac{x}{1+x^2}\, dx$.

For $\int \dfrac{x}{1+x^2}\, dx$, let $w=1+x^2$, then $dw=2x\, dx$.

Therefore $\int \tan^{-1} x\, dx = x\tan^{-1} x - \int \dfrac{x}{1+x^2}\, dx$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{1}{w}\, dw$$
$$= x\tan^{-1} x - \frac{1}{2}\ln|w| + C$$
$$= x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C$$

Note that $\ln\left|1+x^2\right| = \ln\left(1+x^2\right)$ here because $1+x^2>0$ for all real numbers x.

5. Evaluate $\int e^x \cos x \, dx$.

[Solution]

Let $u = e^x$ and $dv = \cos x \, dx$

Then $du = e^x \, dx$ and $v = \sin x$

Thus $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

For $\int e^x \sin x \, dx$

Let $s = e^x$ and $dt = \sin x \, dx$

Then $ds = e^x \, dx$ and $t = -\cos x$

Thus $\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

As a result,
$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx\right] \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx\end{aligned}$$

Therefore, $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C$

[Alternative Solution]

Let $u = \cos x$ and $dv = e^x \, dx$

Then $du = -\sin x \, dx$ and $v = e^x$

Thus $\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$

For $\int e^x \sin x \, dx$

Let $s = \sin x$ and $dt = e^x \, dx$

Then $ds = \cos x \, dx$ and $t = e^x$

Thus $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

As a result,
$$\begin{aligned}\int e^x \cos x \, dx &= e^x \cos x + \int e^x \sin x \, dx \\ &= e^x \cos x + \left[e^x \sin x - \int e^x \cos x \, dx\right] \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx\end{aligned}$$

Therefore, $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C$

6. A particle that moves along a straight line has velocity $v(t)=t^3e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

[Solution]

The particle will move $s(t)=\int_0^t v(x)\,dx$ meters for the first t seconds.

For $\int x^3e^{-x}\,dx$

Let $u=x^3$ and $dv=e^{-x}\,dx$

Then $du=3x^2\,dx$ and $v=-e^{-x}$

Thus $\int x^3e^{-x}\,dx=-x^3e^{-x}+3\int x^2e^{-x}\,dx$

For $\int x^2e^{-x}\,dx$

Let $m=x^2$ and $dn=e^{-x}\,dx$

Then $dm=2x\,dx$ and $n=-e^{-x}$

Thus $\int x^2e^{-x}\,dx=-x^2e^{-x}+2\int xe^{-x}\,dx$

Therefore,
$$\begin{aligned}\int x^3e^{-x}\,dx&=-x^3e^{-x}+3\int x^2e^{-x}\,dx\\&=-x^3e^{-x}+3\left[-x^2e^{-x}+2\int xe^{-x}\,dx\right]\\&=-x^3e^{-x}-3x^2e^{-x}+6\int xe^{-x}\,dx\end{aligned}$$

For $\int xe^{-x}\,dx$

Let $f=x$ and $dg=e^{-x}\,dx$

Then $df=dx$ and $g=-e^{-x}$

Thus $\int xe^{-x}\,dx=-xe^{-x}+\int e^{-x}\,dx$

Therefore,
$$\begin{aligned}\int x^3e^{-x}\,dx&=-x^3e^{-x}-3x^2e^{-x}+6\int xe^{-x}\,dx\\&=-x^3e^{-x}-3x^2e^{-x}+6\left[-xe^{-x}+\int e^{-x}\,dx\right]\\&=-x^3e^{-x}-3x^2e^{-x}-6xe^{-x}+6\int e^{-x}\,dx\\&=-x^3e^{-x}-3x^2e^{-x}-6xe^{-x}-6e^{-x}+C\end{aligned}$$

As a result,
$$\begin{aligned}s(t)&=\int_0^t v(x)\,dx\\&=\left(-t^3e^{-t}-3t^2e^{-t}-6te^{-t}-6e^{-t}+C\right)-\left(-6+C\right)\\&=-t^3e^{-t}-3t^2e^{-t}-6te^{-t}-6e^{-t}+6\end{aligned}$$