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1. The Maclaurin series for e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $-\infty < x < \infty$.

   Evaluate $\lim_{x\to 0} \frac{e^x - 1}{x}$.

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2. The Maclaurin series for $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for $-\infty < x < \infty$.

   Assume the conditions for the **Integral Test** have been verified. Determine the convergence or divergence of the series $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$.

   [Solution]

   Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \geq 1$.

   Note that $0 < \frac{1}{x} \leq 1 < \frac{\pi}{2}$, so $\frac{1}{x}$ is in the first quadrant.

   Hence $f(x)$ is positive for $x \geq 1$.

   In addition, $f(x)$ is continuous for $x \geq 1$.

   Furthermore, $f'(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right) < 0$ for $x \geq 1$.

   Thus $f(x)$ is decreasing for $x \geq 1$.

   Therefore, the Integral Test applies.

   For $x \geq 1$, $\sin\left(\frac{1}{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{x}\right)^{2n+1}$

   $$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{-2n-1}$$

   Thus $\int \sin\left(\frac{1}{x}\right) dx = \int \left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{-2n-1}\right] dx$

   $$= \ln|x| - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!} x^{-2n} + C$$

   Therefore, $\int_1^{\infty} \sin\left(\frac{1}{x}\right) dx = \lim_{t\to\infty} \int_1^t \sin\left(\frac{1}{x}\right) dx$

   $$= \lim_{t\to\infty} \left[\ln|x| - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!} x^{-2n}\right]_1^t$$

   $$= \lim_{t\to\infty} \left[\ln|t| - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!} \frac{1}{t^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!}\right]$$

   Note that $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n+1)!}$ is absolutely convergent by the Ratio Test since

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$$\lim_{x\to\infty}\frac{\frac{1}{(2n+2)(2n+3)!}}{\frac{1}{2n(2n+1)!}}=\lim_{x\to\infty}\frac{2n(2n+1)!}{(2n+2)(2n+3)!}=\lim_{x\to\infty}\frac{n}{(n+1)(2n+3)(2n+2)}=0$$

As a result, $\int_1^\infty \sin\left(\frac{1}{x}\right)\,dx=\lim_{t\to\infty}\left[\ln|t|-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}\frac{1}{t^{2n}}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n(2n+1)!}\right]$

$=\infty$

The improper integral $\int_1^\infty \sin\left(\frac{1}{x}\right)\,dx$ diverges.

Therefore, the infinite series $\sum_{k=1}^{\infty}\sin\left(\frac{1}{k}\right)$ diverges by the Integral Test.