1. Consider the parametric curve Γ :

$$x = t^2$$
$$y = t^3 - 3t$$

a. Find the points on Γ where the tangent is horizontal or vertical.

Answer

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

When $t = 1$ or -1, the tangent is horizontal. When $t = 0$, the tangent is vertical.

$\boxed{\text{The tangent is horizontal at points } (1, -2) \text{ \& } (1, 2). \text{The tangent is vertical at point } (0, 0).}$

b. Determine where Γ is concave up and downward.

Answer

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \\
&= \frac{\frac{d}{dt}(\frac{3t^2-3}{2t})}{2t} \\
&= \frac{2t(6t) - (3t^2 - 3)2}{(2t)^2 2t} \\
&= \frac{12t^2 - 6t^2 + 6}{8t^3} \\
&= \frac{6t^2 + 6}{8t^3} \\
&= \frac{6}{8}\frac{t^2 + 1}{t^3} \\
&= \frac{3}{4}\frac{t^2 + 1}{t^3}
\end{aligned}$$

Therefore, $\boxed{\Gamma \text{ is concave up when } t \text{ is positive and concave downward when } t \text{ is negative.}}$

c. Find the area of the region bounded by Γ and the x-axis. To be more specific, find the area of the region bounded (above) by Γ and bounded (below) by the x-axis.

Answer

It may be helpful to first sketch the graph for a few values of t between $-\sqrt{3}$ and $\sqrt{3}$.

First, we check that $y = 0$ when $t = -\sqrt{3}$, $t = 0$, and $t = \sqrt{3}$.

Next, we find that y is positive when $-\sqrt{3} < t < 0$ and negative when $0 < t < \sqrt{3}$.

Since we want Γ to be above the region and the x-axis to be below our region, we will consider the portion of Γ for $-\sqrt{3} \leq t \leq 0$. Note that $x(-\sqrt{3}) = 3$ and $x(0) = 0$.

The area of our region is

$$\begin{aligned}
\int_0^3 y \text{ dx} &= \int_0^{-\sqrt{3}} (t^3 - 3t)(2t)\text{dt} \\
&= \int_0^{-\sqrt{3}} 2t^4 - 6t^2\text{dt} \\
&= \frac{2}{5}t^5 - \frac{6}{3}t^3\Big|_{t=0}^{t=-\sqrt{3}} \\
&= -\frac{2}{5}9\sqrt{3} + 6\sqrt{3} \\
&= \sqrt{3}\frac{-18+30}{5} \\
&= \sqrt{3}\frac{12}{5}
\end{aligned}$$

<u>Reality check 1 with graphing technology:</u>
My answer is somewhat close to 4. I plot the parametric curve using a graphing tool and roughly estimate that the area of the region is close to 4.

<u>Reality check 2 without computer:</u>
My answer is somewhat close to 4. Using my answer from part (a), I reasoned that the maximum value of y when $-\sqrt{3} < t < 0$ is $y = 2$. Since my region is bounded by $x = 0$ and $x = 3$, and since my upper bound Γ is concave downward, I see that the area should be less than the area of the rectangle with width 2 and height 3.