9.4 Models for Population Growth

Law of natural growth. If $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

$$\frac{dP}{dt} = kP$$

where k is a constant.

Solution to natural growth equation. The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}.$$

Logistic differential equation. The model for population growth known as the logistic differential equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where M is the carrying capacity of P, i.e., the maximum population that the environment is capable of sustaining in the long run.

Solution to the logistic differential equation.

$$P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}.$$

1. **Example:** Suppose a population $P(t)$ satisfies

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

where t is measured in years.

(a) What is the carrying capacity?

(b) What is $P'(0)$?

(c) When will the population reach 50% of the carrying capacity?

Thinking about the problem:

How should I approach this problem? Have I seen a problem similar to this one before? If so, how did I approach it?

I know to find my carrying capacity for part (a), I will need to reformat my differential equation

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

into the form

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where M would be the carrying capacity I am trying to find. I also know that $P'(t) = 0.4P(t) - 0.001P(t)^2$, so using this and the fact that $P(0) = 50$, I can find $P'(0)$ for part (b). For part (c), I can find my function $P(t)$ and solve for $P(t) = \frac{1}{2}M$ that was found in part (a).

Doing the problem:

(a) By factoring, I can find

$$\begin{aligned}\frac{dP}{dt} &= 0.4P - 0.001P^2 \\ &= 0.4P(1 - 0.0025P) \\ &= 0.4P\left(1 - \frac{P}{400}\right)\end{aligned}$$

so the carrying capacity is 400.

(b) Since $P(0) = 50$, I see

$$\begin{aligned}P'(0) &= 0.4P(0) - 0.001P(0)^2 \\ &= 0.4(50) - 0.001(50)^2 \\ &= 20 - 2.5 \\ &= 17.5.\end{aligned}$$

(c) Since the equation can be described as a logistic differential equation, I find

$$A = \frac{M - P_0}{P_0} = \frac{400 - 50}{50} = 7,$$

so the solution is

$$P(t) = \frac{400}{1 + 7e^{-0.4t}}.$$

From part (a), I know the carrying capacity is 400, so 50% of the carrying capacity is 200. So I solve the equation

$$\begin{aligned}
200 &= \frac{400}{1 + 7e^{-0.4t}} \\
1 + 7e^{-0.4t} &= 2 \\
e^{-0.4t} &= \frac{1}{7} \\
-0.4t &= \ln \frac{1}{7} \\
t &= (\ln \frac{1}{7})/(-0.4) \\
&\approx 4.86 \text{ years.}
\end{aligned}$$

Solutions should show all of your work, not just a single final answer.

2. Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10,000. If the population grows to 2500 after one year, what will the population be after another three years?

   (a) What is A?

   (b) What is $P(t)$?

   (c) Using $P(0)$ and $P(1)$, find the constant k from your equation in part (b).

   (d) Find $P(4)$.

3. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.

   (a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

   (b) How long will it take for the population to increase to 5000?

4. T/F (with justification): The solution of the initial-value problem $\frac{dP}{dt} = kP$ and $P(0) = P_0$ is $P(t) = P_0 e^{kt}$ (requires <u>justification</u>, not just quoting the textbook).