9.3 Separable Equations

Separable equation. A separable equation is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

Orthogonal trajectory. An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles. Thus if the derivative of a family of curves is $\frac{dy}{dx}$, then the derivative of the orthogonal trajectory would be $-\frac{1}{\frac{dy}{dx}}$.

Note: Two curves are orthogonal, that is, meet at right angles provided that the slopes of the curves are negative reciprocals of each other.

Mixing problems. If $y(t)$ denotes the amount of substance in the tank at time t, then

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}).$$

1. **Example:** A tank contains 1000L of brine with 15kg of dissolved salt. Pure water enters the tank at a rate of 10L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after t minutes? After 20 min?

   Thinking about the problem:

   How should I approach this problem? Have I seen a problem similar to this before? If so, how did I approach it?

This is a mixing problem, so I know that I will need to find the rate in and rate out in order to find

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}).$$

I know that rate in is equal to the concentration multiplied by the rate of liquid entering the tank. Similarly, the rate out is the concentration of salt multiplied by the rate of liquid leaving the tank. Let $y(t)$ be the amount of salt in the tank. Once I find $\frac{dy}{dt}$, I have an initial-value problem. This will give me the salt in the tank after t minutes. To find the salt in the tank after 20 min, I need to let $t = 20$ and find $y(20)$.

Doing the problem:

From the problem, I know that pure water enters at a rate of 10L/min and the mixture drains from the tank at the same rate, i.e., 10 L/min. Let $y(t)$ be the amount of salt in the tank. I know that $y(0) = 15\text{kg}$ since initially, the "tank contains 1000L of brine with 15kg of dissolved salt" as stated in the problem. The problem also states, "pure water enters the tank at a rate of 10L/min" but pure water contains no salt. So the concentration of salt is 0, so the rate of salt entering the tank is 0. Now we know

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = 0 - (\text{rate out}) = -(\text{rate out}).$$

The concentration of salt that is leaving the tank should be

$$\frac{\text{salt in tank}}{\text{volume of tank}} = \frac{y(t)\text{kg}}{1000\text{L}},$$

so the rate of salt flowing out of the tank is

$$\text{concentration} \cdot \text{rate of liquid leaving the tank} = \frac{y(t)\text{kg}}{1000\text{L}} \cdot 10\text{L/min}.$$

Therefore

$$\frac{dy}{dt} = -\frac{y}{100}.$$

I can solve this differential equation using the method of separable equations and find

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{y}{100} \\
\int \frac{dy}{y} &= -\frac{1}{100}\int dt \\
\ln y &= -\frac{t}{100} + C
\end{aligned}$$

and since $y(0) = 15$

$$\begin{aligned}
\ln y(0) &= \frac{0}{100} + C \\
\ln 15 &= C.
\end{aligned}$$

So,

$$\begin{aligned}
\ln y &= \ln 15 - \frac{t}{100} \\
\ln y - \ln 15 &= -\frac{t}{100} \\
\ln\left(\frac{y}{15}\right) &= -\frac{t}{100} \\
\frac{y}{15} &= e^{-t/100} \\
y &= 15e^{-t/100}.
\end{aligned}$$

Therefore, the amount of salt in the vat at time t is $y(t) = 15e^{-t/100}$kg. The amount of salt after 20 minutes is $y(20) = 15e^{-20/100} \approx 12.3$kg.

Solutions should show all of your work, not just a single final answer.

2. A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

   (a) Let $y(t)$ be the amount of alcohol in the vat after t minutes. What is $y(0)$?

   (b) What is the rate the alcohol is entering the vat?

   (c) What is the rate the alcohol is leaving the vat?

   (d) What is $\dfrac{dy}{dt}$?

   (e) Solve the separable equation from (d).

   (f) Find the <u>percentage</u> of alcohol after one <u>hour</u> (what should t be?).

3. A tank contains 1000L of pure water. Brine that contains .005 kg of salt per liter of water enters the tank at a rate of 5L/min. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15L/min. How much salt is in the tanks after t minutes? After one hour?

4. Find the solution to $\dfrac{dy}{dx} = e^x e^y$ where $y(0) = 1$.

5. Solve the differential equation.

$$(y^2 + xy^2)y' = 1$$

6. Find the orthogonal trajectories of the family of curves.

$$y^2 = kx^3$$

7. T/F (with justification)

   The differential equation $\dfrac{dy}{dx} = yx + y$ is separable.