9.1 Modeling with Differential Equations

Differential equation (DE). An equation that contains an unknown function and some of its derivatives. Some examples of DE include $\frac{dy}{dx} = 4x$, $\frac{dy}{dx} = 2x^2 - 4$ and $\frac{d^2y}{dx^2} = 5x - 1$. The order of a DE is the order of the highest order derivative that appears in the equation. For example, $\frac{dy}{dx} = 4x$ is a first order DE while $\frac{d^2y}{dx^2} = 5x - 1$ is a second order DE.

1. **Example:** Which of the following functions are solutions to the differential equation $y'' + y = \sin x$?

   (a) $y = \sin x$

   (b) $y = \cos x$

   (c) $y = \frac{1}{2}x \sin x$

   (d) $y = -\frac{1}{2}x \cos x$

   Thinking about the problem:

   How should I approach this problem? Have I seen a problem like this before? If so, how did I approach it?

   I notice that this problem is asking me to check if different functions are solutions to the differential equation $y'' + y = \sin x$. So first, I will have to find y and y'' for each of the following functions.

   (a) $y = \sin x$

   (b) $y = \cos x$

   (c) $y = \frac{1}{2}x \sin x$

(d) $y = -\frac{1}{2}x\cos x.$

I will need to find y'' and plug both y and y'' into the equation $y''+y = \sin x$ to determine if it is a solution or not.

Doing the problem:

First, I find y in each of the functions is

(a) $y = \sin x$

(b) $y = \cos x$

(c) $y = \frac{1}{2}x\sin x$

(d) $y = -\frac{1}{2}x\cos x.$

Next, I find y'' in each of the functions is

(a) $y'' = -\sin x$

(b) $y'' = -\cos x$

(c) $y'' = \cos x - \frac{1}{2}x\sin x$

(d) $y'' = \sin x + \frac{1}{2}x\cos x.$

So $y'' + y$ in each function is

(a) $-\sin x + \sin x = 0$

(b) $-\cos x + \cos x = 0$

(c) $y = \cos x - \frac{1}{2}x\sin x + \frac{1}{2}x\sin x = \cos x$

(d) $y = \sin x + \frac{1}{2}x\cos x - \frac{1}{2}x\cos x = \sin x.$

So the only solution to $y'' + y = \sin x$ is the function (d) $y = -\frac{1}{2}x\cos x$.

Solutions should show all of your work, not just a single final answer.

2. We consider the differential equation $\frac{dy}{dt} = 1 - 2y$.

(a) Find all constant solutions (*Hint:* Check functions $y = K$ for constant K. What does K need to be for the function to be a solution to the differential equation?).

(b) Show every function of the form $y(t) = \frac{1}{2} + Ke^{-2t}$, where K is a constant, is a solution.

3. We consider the differential equation $\frac{dy}{dx} = xy$.

   (a) Find all constant solutions.

   (b) Show every function of the form $y(x) = Ke^{x^2/2}$, where K is a constant, is a solution.

4. T/F (with justification)

   Every differential equation has a constant solution.