7.3 Trigonometric Substitution

In each of the following trigonometric substitution problems, draw a triangle and label an angle and all three sides corresponding to the trigonometric substitution you select.

Table of Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ OR $x = a\cos\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ OR $x = a\cot\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ OR $x = a\csc\theta$	$1 + \tan^2\theta = \sec^2\theta$

1. **Example:** Evaluate $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}$.

   Thinking about the problem:

   What technique of integration should I use to evaluate the integral and why? Have I seen a problem similar to this one before? If so, which technique did I use?

   I know that I need to evaluate an indefinite integral, so I know my answer will include $+C$. To determine which technique to use, I will focus on the integrand, i.e., $\dfrac{1}{\sqrt{9 - x^2}}$.

   I think I will use the technique of Trigonometric Substitution to evaluate this integral. I noticed that the denominator of the integrand is $\sqrt{9 - x^2}$, which is a form found in the Table of Trigonometric Substitution. I look at the table and find that the substitution

I want to use is $x = 3\sin\theta$. I can then simplify my integral with this substitution and integrate.

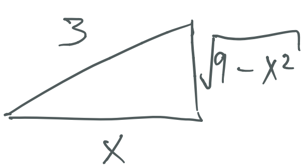
Doing the problem:

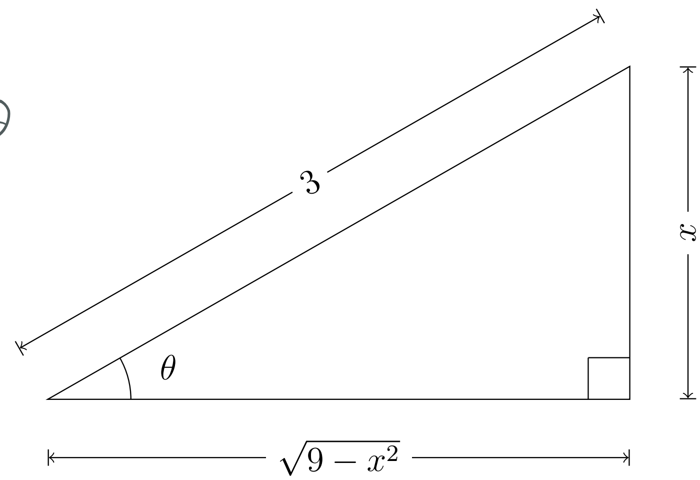
I will evaluate $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$ using the technique of Trigonometric Substitution. I will let $x = 3\sin\theta$ and $dx = 3\cos\theta\, d\theta$. Then I can draw a triangle using my choice of substitution and find the following picture:

OR

$x = 3\cos\theta$

$dx = -3\sin\theta\, d\theta$





So the new integral is

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{\sqrt{9-(3\sin\theta)^2}}$$

$$= \int \frac{3\cos\theta\, d\theta}{\sqrt{9(1-\sin^2\theta)}}$$

$$= \int \frac{3\cos\theta\, d\theta}{\sqrt{9\cos^2\theta}}$$

$$= \int \frac{3\cos\theta\, d\theta}{3\cos\theta}$$

$$= \int d\theta$$

$$= \theta + C.$$

Since the substitution we used was $x = 3\sin\theta$, then $\theta = \sin^{-1}\left(\frac{x}{3}\right)$. So

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C.$$

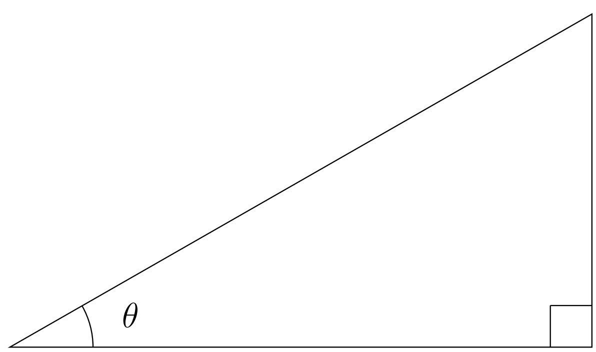
Solutions should show all of your work, not just a single final answer.

2. Evaluate $\displaystyle\int \frac{dx}{(9+x^2)^{3/2}}$.

(a) State the technique of integration you would use to evaluate the integral.

(b) Which substitution would you use for x? What would dx be?

(c) Based on the choice for x, fill in the triangle:



(d) Using (b), write the new integral.

(e) Evaluate (optional - do at home):

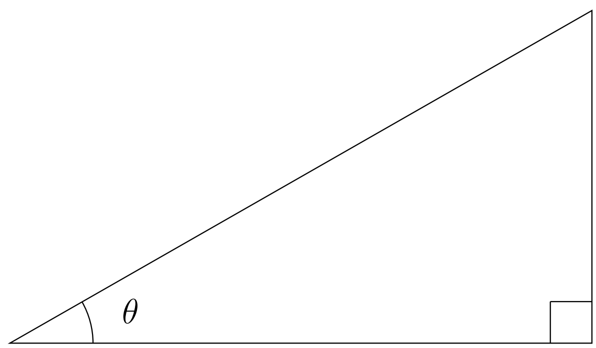
Answer : $\frac{1}{9}\frac{x}{\sqrt{3^2+x^2}}$ + Constant

3. Evaluate $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$.

(a) State the technique of integration you would use to evaluate the integral.

(b) Which substitution would you use for x? What would dx be?

(c) Based on the choice for x, fill in the triangle:



(d) Using (b), write the new integral.

(e) Using (c) and (d), evaluate the integral. (optional – work at home)

Ans: $\frac{1}{6}\left[\operatorname{arcsec}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{x^2}\cdot 3\right]$

4. Evaluate $\displaystyle\int_0^3 \frac{x^2}{\sqrt{9-x^2}}\,dx$. (*Hint:* When you make a trigonometric substitution, include the bounds of integration in the substitution.)

5. T/F (with justification): To evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+2}}$ by trigonometric substitution, use $x = 2\tan\theta$.