7.1 Integration by Parts

Integration By Parts.
Integrals in the form of $\int u\,dv$ can be solved using the formula $\int u\,dv = uv - \int v\,du$.

1. **Example**: Evaluate $\int x^2 e^x\,dx$.

   Thinking about the problem:

   What technique of integration should I use to compute the integral and why? Have I seen a problem similar to this one before? If so, which technique did I use?

   I know that I need to compute an indefinite integral, so I know my answer will include $+C$. To determine which technique I should use, I will focus on the integrand, i.e., x^2e^x. I think I can use the Integration by Parts (IBP) technique in this case. The formula for IBP is given by: $\int u\,dv = uv - \int v\,du$; so to use this technique, I will need expressions for u, v, du, and dv. I will take $u = x^2$, which leaves $dv = e^x$. From these substitutions, I can find du and v and substitute these expressions into the formula for IBP. I need to check the integral found in the formula when I apply the IBP technique. If this new integral is more difficult to solve than the original integral then I probably made a mistake in my assignment of u and dv. Finally, I may need to use IBP again in the new integral found by applying the IBP technique.

Doing the problem:

The problem asks me to compute the integral. I choose to apply the IBP technique. I know the integrand $u\,dv = x^2e^x$, and I define $u = x^2$ and $dv = e^x$. Next, I can compute $du = 2x$ and $v = e^x$ and put my results in a table.

$u = x^2$	$dv = e^x$
$du = 2x$	$v = e^x$

I can now use the formula $\displaystyle\int u\,dv = u\,v - \int v\,du$ to find $\displaystyle\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$.

At this point, I notice that I must use IBP again on $\displaystyle -\int 2xe^x\,dx = \int -2xe^x\,dx$. So, I can complete the following table:

$u = -2x$	$dv = e^x$
$du = -2$	$v = e^x$

Using the formula, I can compute $\displaystyle\int -2xe^x\,dx = -2xe^x - \int(-2e^x)\,dx = -2xe^x + \int 2e^x\,dx$. I can evaluate the integral $\displaystyle\int 2e^x\,dx = 2e^x + C$. Therefore, I put together the pieces to find

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - \int 2xe^x\,dx \\ &= x^2e^x + \int(-2xe^x)\,dx \\ &= x^2e^x - 2xe^x + \int 2e^x\,dx \\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned}$$

2. Evaluate $\int x^2 e^{-3x}\, dx$.

   (a) State the technique of integration you would use to compute the integral.

   (b) Compute the following table:

$u =$	$dv =$
$du =$	$v =$

3. Evaluate $\int_0^{\pi} x\cos(3x)\,dx$.

   (a) State the technique of integration you would use to evaluate the integral.

   (b) Complete the following table:

$u =$	$dv =$
$du =$	$v =$

   (c) Compute the new integral found after applying the Integration By Parts technique (Is this integral more difficult to solve than the original integral?)

   (d) Evaluate $\int x\cos(3x)dx$.

   (e) Using part (d), evaluate $\int_0^{\pi} x\cos(3x)dx$.

4. Evaluate $\displaystyle\int_0^{\pi} x^2 \sin x \, dx$.

5. Evaluate $\int e^x \sin(x)dx$ (*Hint:* You may need to think of the expression as an equation and add or subtract an integral to both sides of the equation).

6. True or False (Provide a justification for your answer). For differentiable $f(x)$,

$$\int_0^{\pi} f(x)\cos x\,dx = -\int_0^{\pi} f'(x)\sin x\,dx.$$