11.11 Applications of Taylor Polynomials

Taylor polynomial of f at a. The nth degree Taylor polynomial of a function f at $x = a$ is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor's Inequality. If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \qquad |x-a| \leq d.$$

1. **Example:** Determine the 3rd-degree Taylor polynomial $T_3(x)$ for $\sqrt{x}$ at $x = 4$ and use Taylor's inequality to estimate the error $|\sqrt{x} - T_3(x)|$ if $|x-4| \leq .5$.

   Thinking about the problem:

   What formulas should I use to determine the Taylor polynomial and to estimate the error and why? Have I seen a problem similar to this before? If so, what formulas did I use?

   To determine the 3rd-degree Taylor polynomial at for $\sqrt{x}$ at $x = 4$, I will need to recall that

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.$$

I will make a table to determine all of the necessary coefficients. Then I can plug in my coefficients into the equation to find $T_3(x)$. To estimate the error, I will need to find M so that $|f^{(4)}(x)| \leq M$ and use Taylor's Inequality.

Doing the problem:

To find $T_3(x)$, I begin by making the following table:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{256}$

By the formula for $T_3(x)$, I find

$$\begin{aligned} T_3(x) &= 2 + \frac{1}{4}(x-4) - \frac{1}{32} \cdot \frac{1}{2!}(x-4)^2 + \frac{3}{256} \cdot \frac{1}{3!}(x-4)^3 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3. \end{aligned}$$

To use Taylor's Inequality to find the upper bound of the error of $T_3(x)$ on $|x-4| < .5$, I need to find M such that $|f^{(4)}(x)| \leq M$. I find

$$|f^{(4)}(x)| = \left| -\frac{15}{16}x^{-7/2} \right| \leq \frac{15}{16}(3.5)^{-7/2} \qquad \text{on } |x| < .5.$$

So

$$R_3(x) \leq \frac{M}{4!}|x-4|^4 \leq \frac{15}{16}(3.5)^{-7/2} \cdot \frac{1}{4!}(.5)^4 \approx .00001262979,$$

which is an upper bound to the error $|\sqrt{x} - T_3(x)|$ on $|x-4| \leq .5$.

Solutions should show all of your work, not just a single final answer.

2. Determine the 4th-degree Taylor polynomial $T_4(x)$ for $f(x) = \dfrac{1}{(x-5)^2}$ at $x = 1$ and use Taylor's inequality to estimate the error $|f(x) - T_4(x)|$ if $|x - 1| \leq .5$.

(a) Fill in the following table

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0		
1		
2		
3		
4		

(b) Determine the 4th-degree Taylor polynomial for $\dfrac{1}{(x-5)^2}$ at $x = 1$.

(c) Find M such that $|f^{(5)}(x)| \leq M$ for $|x - 1| \leq .5$.

3. Use Taylor's inequality to determine a partial sum for the Maclaurin series of $\cos x$ (with x in radians) that is within .0001 of $\cos 2$.

4. T/F (with justification)

   The 2nd-degree Taylor polynomial at 0 for $\sqrt{1+x}$ is $1 + (1/2)x - (1/4)x^2$.