11.10 Taylor and Maclaurin Series

Taylor series of the function f at a. The Taylor series of the function f at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin Series. For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

This case arises frequently enough that it is given the special name <u>Maclaurin series</u>.

Important Maclaurin Series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1$$

1. **Example:** Compute the Taylor series for the function $f(x) = x^2 + 3x - 4$ at $a = 1$ and compute the Maclaurin series for the same function.

   Thinking about the problem:

   Have I seen a problem similar to this one before? If so, what did I do to compute the Taylor series?

   To determine the Taylor series for the function $f(x)$, I will make a table with n, $f^{(n)}(x)$, and $f^{(n)}(a)$ for enough n to determine a pattern. Then I will use this pattern to find the series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

   Doing the problem:

   The problem asks to find the Taylor series for the function $f(x) = x^2 + 3x - 4$ at $a = 1$ as well as the Maclaurin series for $f(x)$. This means I will find the Taylor series for $f(x)$ at $a = 1$ and the Taylor series of $f(x)$ at $a = 0$ (i.e., Maclaurin series). So I will make a table as follows:

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(0)$
0	$x^2 - 3x - 4$	$1 - 3 - 4 = -6$	-4
1	$2x - 3$	$2 - 3 = -1$	-3
2	2	2	2
3	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	0	0

   The Taylor series of f at $a = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = -6 - (x-1) + (x-1)^2$$

and the Taylor series of f at $a = 0$ (i.e., Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = -4 - 3x + x^2.$$

Solutions should show all of your work, not just a single final answer.

2. Let $f(x) = \sqrt{x}$.

   (a) Can you write a Maclaurin series for f? Explain why or why not.

   (b) Fill in the following table:

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0		
1		
2		
3		
4		

   (c) Using the table above write out the Taylor Series for $f(x) = \sqrt{x}$ at $a = 9$.

3. Use the formulas for e^x and $\arctan x$ to find the Taylor series centered at 0 for the following functions. Specify the radius of convergence of each new series.

   (a) $f(x) = e^{3x}$

   (b) $f(x) = \arctan(x/3)$

4. T/F (with justification)
   If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \cdots$ for $|x| < 1$ then $f'''(0) = 30$.