10.3 Polar Coordinates

Polar Coordinates/Cartesian Coordinates.

1. A) To find the Cartesian coordinates of a point when the polar coordinates are known, use:

$$x = r\cos\theta \qquad y = r\sin\theta$$

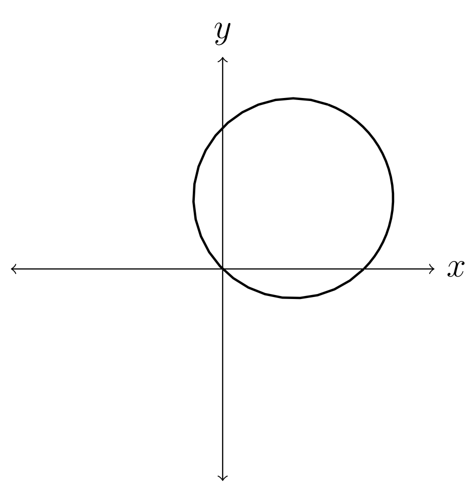
2. B) To find the polar coordinates of a point when the Cartesian coordinates are known, use:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Derivatives.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\left(\frac{dr}{d\theta}\cdot\sin\theta\right) + r\cos\theta}{\left(\frac{dr}{d\theta}\cdot\cos\theta\right) - r\sin\theta}$$

1. **Example:** Here is the plot of the polar equation $r = \sin\theta + \cos\theta$.



Fill in the table below and find the equation of the tangent line to the curve at $(x, y) = (0, 0)$.

θ	$\sin\theta + \cos\theta$	(r, θ)	(x, y)
0			
$\pi/2$			
π			
$3\pi/2$			

Thinking about the problem:

How should I approach this problem? What is it asking me to do? Have a seen a problem similar to this one before? If so, what did I do?

The first part of the question is asking me to fill in a table for values of $r = \sin\theta + \cos\theta$, (r, θ), and (x, y). Using the formulas $x = r\cos\theta$ and $y = r\sin\theta$, I can compute the values in the table. The second part of the question is asking me to find the equation of the tangent line to the curve at $(x, y) = (0, 0)$. First, I will need to find for what value θ does $(x, y) = (0, 0)$. Then I can find the derivative using the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\left(\frac{dr}{d\theta} \cdot \sin\theta\right) + r\cos\theta}{\left(\frac{dr}{d\theta} \cdot \cos\theta\right) - r\sin\theta}.$$

Finally, I will recall that the equation of a tangent line at the point (x_0, y_0) is

$$y - y_0 = \frac{dy}{dx}(x - x_0).$$

I can then find the information I need to find the equation of the tangent line.

Doing the problem:

I can compute the values in the table using the formula $x = r\cos\theta$ and $y = \sin\theta$ to find the following:

θ	$\sin\theta+\cos\theta$	(r,θ)	(x,y)
0	1	$(1,0)$	$(1,0)$
$\pi/2$	1	$(1,\pi/2)$	$(0,1)$
π	-1	$(-1,\pi)$	$(1,0)$
$3\pi/2$	-1	$(-1,3\pi/2)$	$(0,1)$

I find that $(x,y)=(0,0)$ is when $r=\sin\theta+\cos\theta=0$ so $\sin\theta=\cos\theta$, which is true when $\theta=\frac{\pi}{4}$. To find the equation of the tangent line, I find

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$=\frac{\left(\frac{dr}{d\theta}\cdot\sin\theta\right)+r\cos\theta}{\left(\frac{dr}{d\theta}\cdot\cos\theta\right)-r\sin\theta}$$

In my formula alone I need to find $\frac{dr}{d\theta}$. Since $r=\sin\theta+\cos\theta$ then $\frac{dr}{d\theta}=\frac{d}{d\theta}(\sin\theta+\cos\theta)=\cos\theta-\sin\theta$.

$$\frac{dy}{dx}=\frac{(\cos\theta-\sin\theta)\cdot\sin\theta+(\sin\theta+\cos\theta)\cdot\cos\theta}{(\cos\theta-\sin\theta)\cdot\cos\theta-(\sin\theta+\cos\theta)\cdot\sin\theta}$$

$$=\frac{\cos\theta\cdot\sin\theta-\sin^2\theta+\sin\theta\cdot\cos\theta+\cos^2\theta}{\cos^2\theta-\sin\theta\cdot\cos\theta-\sin^2\theta-\cos\theta\cdot\sin\theta}$$

$$=\frac{\cos^2\theta-\sin^2\theta+2\cos\theta\sin\theta}{\cos^2\theta-\sin^2\theta-2\sin\theta\cos\theta}$$

Letting $\theta = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right) - 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)}$$

$$= \frac{\frac{1}{2} - \frac{1}{2} + 2 \cdot \frac{1}{2}}{\frac{1}{2} - \frac{1}{2} - 2 \cdot \frac{1}{2}}$$

$$= -1$$

So I can conclude that the tangent line is

$$y - y_0 = \frac{dy}{dx}(x - x_0)$$

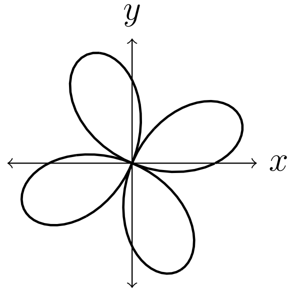
$$y = -x.$$

Solutions should show all of your work, not just a single final answer.

2. Give two additional representations of P in polar coordinates:

$$\text{(a)}\quad P = \left(2, \frac{\pi}{4}\right), \qquad \text{(b)}\quad P = \left(-1, -\frac{\pi}{3}\right).$$

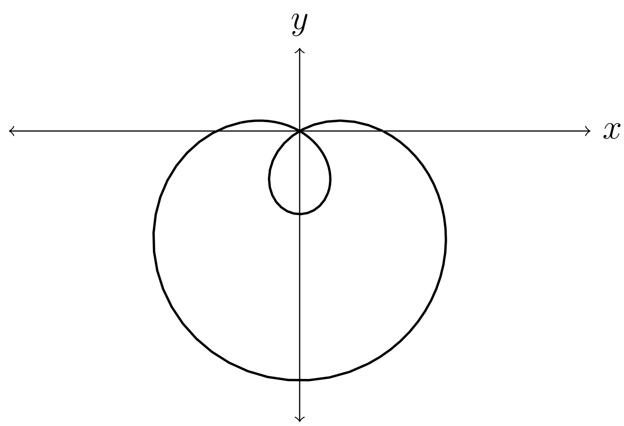
3. Below is a graph of the polar equation $r = \sin(2\theta) + \cos(2\theta)$. Fill in the table below for this polar graph.



θ	$\sin(2\theta) + \cos(2\theta)$	(r, θ)	(x, y)
0			
$\pi/2$			
π			
$3\pi/2$			
2π			

Based on this table, and additional data if it seems needed, draw arrows on the curve (including on each loop) to indicate the direction of θ as it increases from 0 to 2π.

4. Below is a plot of the polar equation $r = 1 - 2\sin\theta$.



Find the equation of the tangent line to the curve $r = 1 - 2\sin\theta$ at $(x, y) = (1, 0)$.

5. T/F (with justification)

Every point in the plane besides the origin can be written in polar coordinates (r, θ) with $r < 0$.