(i) You are encouraged to discuss your work with your classmates but this assignment is to be completed individually (i.e, don't show anyone your full solutions). (ii) Check your work with other methods or with the computer whenever possible. (iii) After you've done some work, you may to ask for more hints during class, office hours, or on Piazza.

1. Name ______________________________

   (Credit) This cannot be left blank. If you didn't talk to anyone or read other sources, please give an explanation.

   Please give credit to people that you talked with (including the people that *you* helped), any textbooks you used (other than Stewart), and the website addresses of any resources you have used.

2. (Style points) You will earn full "Style points" if each of your solution is legible, coherent, and not ambiguous. Your reader should not need to reread your solution several times to find a train of thought. Your final draft should not include any scratch work. In addition, you should use correct mathematical notations. This includes not writing an equal sign between two unequal objects and not treating the symbol ∞ like a number (for example, don't attempt to multiply 0 with the symbol ∞).

3. For the first two parts of this problem, suppose $f(x)$ is continuous everywhere.

   (a) True of false? $\displaystyle\int_0^a f(x) = \int_0^a f(x-a)$ dx. If true, justify. If false, give a counterexample. (Hint: use u-substitution $u = x - a$.)

   (b) True of false? $\displaystyle\int_0^a f(x) = \int_0^a f(a-x)$ dx. If true, justify. If false, give a counterexample. (Hint: use u-substitution $u = a - x$.)

(c) Use the identity $\sin(\pi/2 - A) = \cos(A)$ to explain that $\sin(\pi/2 - x) = \cos(x)$ and $\sin(x) = \cos(\pi/2 - x)$.

(d) Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\,\mathrm{dx} = \frac{\pi}{4}.$$

(Hint 2: Note that $\sin(x)$ is continuous everywhere.)
(Hint 1: Instead of computing this directly, use the identities mentioned in the first three parts of this problem to first show that this definite integral is equal to $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\mathrm{dx}$.

4. Let $\Gamma(X)$ be a function on $(0, \infty)$ defined by

$$\Gamma(X) := \int_{t=0}^{\infty} t^{X-1} e^{-t} \text{ dt}$$

This amazing function shows up a lot in analysis (like Calculus), probability, and combinatorics. (Note: Γ is the upper case symbol of the 3rd Greek letter, pronounced "Gamma" in English)

(a) Evaluate $\Gamma(2)$

(b) Evaluate $\Gamma(3)$

(c) Evaluate $\Gamma(4)$

(d) Use the computer to compute $\Gamma(5)$, $\Gamma(6)$, and $\Gamma(7)$. What does this remind you of?

(e) Use the fact that $\int_0^\infty e^{-x^2}\,\mathrm{dx} = \frac{1}{2}\sqrt{\pi}$ to compute $\Gamma(1/2)$.
(Note: this is just one of the cool properties this function $\Gamma(X)$ has).

5. The federal government uses a principle called the *multiplier effect* to justify deficit spending. According to this theory, a single dollar of government spending might increase total economic output by more than 1 dollar: When money is spent on goods and services, those who receive the money also spend some of it. The people receiving the money will spend some of that, and so on. This chain reaction is called the *multiplier effect* in economics.

   In a hypothetical isolated community, the process begins when the government spends D dollars. The people who receive it spends a fraction c of those D dollars (that is, Dc dollars). Those who receive the Dc dollars spend a fraction c of it (that is, Dc^2 dollars), and so on. The value c is called the *marginal propensity to consume* in economics. Assume that $0 < c < 1$ because in this fictional community you can only spend less than 100% of what you have received.

   Optional: Watch the following video/s if you don't understand the questions (written below) OR/and if you are interested in economics.

   Khan Academy's "Marginal Propensity to Consume" video (less-mathy version): `https://www.khanacademy.org/economics-finance-domain/macroeconomics/income-and-expenditure-topic/mpc-tutorial/v/mpc-and-multiplier`

   Khan Academy's "Marginal Propensity to Consume" video (with proof of the formula for partial sum of geometric series): `https://www.khanacademy.org/economics-finance-domain/macroeconomics/income-and-expenditure-topic/mpc-tutorial/v/mathy-version-of-mpc-and-multiplier-optional`

   (a) Let S_n be the total spending that has been generated after n transactions, that is, $S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1}$. Show all computation for finding the formula for S_n and write down the formula for S_n. (Hint: see the explanation for eq. 3 on page 709 or watch the second video above).

(b) Suppose that each recipient of spent money saves s of the money that they receive. This value s is called the *marginal propensity to save* in economics. (Note: each person spends c of the money the receive and saves s of the money they receive, so $c + s = 1$). Compute

$$\lim_{n\to\infty} S_n$$

and write it in terms of s and D. (Hint: see the solution to Sec 11.2, Example 2 on page 709)

(c) Let $k := \dfrac{1}{s}$. This number k is called the *multiplier* in economics. What is the multiplier if the marginal propensity to consume is 60%? (Hint: watch the above video/s if you don't know how to compute the answer).

6. Recall the Fibonacci sequence defined by the recurrence relation

$$F_1 = 1, F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3.$$

(a) Use the recurrence relation definition to show that

$$\frac{1}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}$$

(b) Compute $\displaystyle\sum_{n=2}^{N} \frac{1}{F_{n-1}F_{n+1}}$ (Hint: use part (a) to rewrite this as a telescoping sum)

(c) Use your answer above to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}}$ is convergent or divergent. If it is convergent, compute the infinite sum. (You may use the fact that $\lim_{n\to\infty} F_n = \infty$).

(d) Compute $\displaystyle\sum_{n=2}^{N} \frac{F_n}{F_{n-1}F_{n+1}}$ (Hint: use part (a) to rewrite this as a telescoping sum)

(e) Use your answer above to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}}$ is convergent or divergent. If it is convergent, compute the sum (You may use the fact that $\lim_{n\to\infty} F_n = \infty$).

7. The meaning of the decimal expansion of a number

$$0.d_1d_2d_3\dots$$

(where the digit d_i is one of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$, or 9) is that

$$0.d_1d_2d_3\dots = \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \dots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}.$$

Show that this series always converges (which explains why we can write infinite decimal expansions).

(Hint: you may want to use the comparison test as well as the geometric series test)

8. For each statement below, justify why it is true or false.
(Hints: see Sec 11.8: Thm 4, top of pg 749; table at the bottom of pg 749; Example 5 pg 750).

(a) It is possible to find a power series whose interval of convergence is $[0, \infty)$.
(Give an example or explain why it's false.)

(b) It is possible to find a power series whose interval of convergence is $(-\infty, \infty)$.
(Give an example or explain why it's false.)

(c) It is possible to find a power series whose interval of convergence is $\{5\}$.
(Give an example or explain why it's false.)

(d) It is possible to find a power series whose interval of convergence is $(0, 1]$.
(Give an example or explain why it's false.)

(e) It is possible to find a power series whose interval of convergence is $(0, 1)$.
(Give an example or explain why it's false.)

(f) It is possible to find a power series whose interval of convergence is $\{0, 1\}$.
(Give an example or explain why it's false.)

9. (Write your own problem) Review the most recent topics from chapter 11. The solution should not require topics beyond Calculus II materials. A few questions submitted from students may be chosen for future questions. Past submitted problems are currently listed here: `https://www.overleaf.com/read/wksjxzxthbvm`

   (a) Explain briefly which concepts you wish to highlight and why (for example, maybe it's a brand-new concept to you or has applications to science).

   (b) Write a new problem. *(Do not write an answer key on the same page).* It should require a decent understanding of one or more of the concepts/ skills to solve. It should not be too similar to the problems your study mates submitted.

   (c) i. Ask a couple students from our class to solve your problem without showing them the solution. If they are having trouble solving it, explain the solution. Write the names of the people who have attempted to solve your problem.

   ii. Write a brief, one-sentence comment about what other people thought (ask them to give a comment more constructive than 'good job' or 'this is a difficult problem', etc).

(Please go to the next page)

(d) Write a complete answer key to your problem.