Math 1152Q Problems A (KEY) Fall 2017

Please inform me of typos

Week 3 Wednesday, September 13

You are encouraged to discuss your work with your classmates but this assignment is to be completed individually (that is, don't share/ let someone take a picture of your full solutions). Solutions must be written on separate paper(s) with each problem clearly marked. Show all work neatly and give thoughtful explanations where necessary.

1. (Credit) Please give credit to people who worked with you (including the people that *you* helped and the people who gave you help), any textbooks or notes you used (other than Stewart), and the website addresses of any internet resources you might have used (other than WebAssign).

2. (Style points) You will earn full "Style points" if each of your solution is legible, coherent, and not ambiguous. In addition, your reader should not need to reread your solution several times to find a train of thought. One tip is that your final draft should not include any scratch work. (I expect almost everyone to receive full points on this - you can show me a sample of your work before the due date if you are not sure).

3. (Key words: vocabulary, Sec 2.5, continuity, interval)

   (a) (copy from page 115) Let f be a function and let r be a real number. According to Stewart's definition, f is called *continuous* at r if: $f(r)$ is ____________ , ____________ exists, and $f(r) =$ ____________.

   Warning: a function does not need a derivative at r to be continuous at r.

   (b) (copy from page 117) Let I be the open interval (a, b), that is, $\{x$ is a real number : $a < x < b\}$. What does it mean for f to be continuous on the interval I?

4. (Key words: vocabulary, limits and sequences, convergent series)

   (a) (copy from Sec 11.1 page 696) Let $\{a_n\}$ be a sequence and let $L \in \mathbb{R}$ (this notation means that L is a real number). What does $\lim_{n\to\infty} a_n = L$ mean? Warning: do not include variations of the words "converge", "diverge", "approach", or "infinity" in your answer. Another warning: in general $\lim_{n\to\infty} a_n = L$ does *NOT* imply that "a_n will never reach L".

   (b) (copy from Sec 11.1 page 697) What does it mean to write $\lim_{n\to\infty} b_n = \infty$? Warning: do not include variations of the words "converge", "diverge", "approach", "increase", "continuously", or "infinity" in your answer.

   (c) (copy from Sec 11.2, page 708) Let $\{c_n\}$ be a sequence. What is a partial sum of $\{c_n\}$?

   (d) (copy from Sec 11.2, page 708) Let $\{c_n\}_{n=1}^{\infty}$ be a sequence. We say that the infinite series $\sum_{n=1}^{\infty} c_n$ is *convergent* if

   ____. (Hint: your answer should include the words 'limit' and 'partial sums') If the above blank is not true, then we say that $\sum_{n=1}^{\infty} c_n$ is *not convergent* or *divergent*.

5. (Key words: Sec 11.1, monotonic sequence theorem, page 702) Either give an example of a sequence satisfying the condition (no explanation necessary) or explain why no such sequence exists.

(a) A monotonically decreasing sequence that converges to 10.

ANSWER (possible answers): $\left\{10+\frac{1}{n}\right\}_{n=1}^{\infty}$ or $\left\{10+\frac{1}{2^n}\right\}_{n=1}^{\infty}$

(b) A monotonically increasing bounded sequence that does not converge (that is, diverges).

ANSWER: No such sequence exists because the monotone sequence theorem says: *if a sequence is monotonically increasing and bounded, then it converges.*

(c) A non-monotonic sequence that converges to $\frac{3}{4}$.

ANSWER (a possible answer): $\left(\frac{-1}{2}\right)^n+\frac{3}{4}$ converges to $0+\frac{3}{4}$ and is not monotonic.

6. (Key words: Sec 11.1, precise definition) Each of the following series a_n converges to a limit L. For $\epsilon = \frac{1}{100}$, find *a* positive number N such that, if $n > N$, then a_n is within distance ϵ of L. Then for an arbitrarily small ϵ, find *an* N. SEE EXAMPLE `https://egunawan.github.io/fall17/notes/notes11_1choosingN.pdf`

(a) $a_n = \frac{1}{n^2+3}$, $L = 0$.

ANSWER (a possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \sqrt{\frac{1}{\epsilon}}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{1}{k^2+3} - 0\right| \\
&= \frac{1}{k^2+3} \\
&< \frac{1}{N^2+3} \text{ since } k > N \text{ implies that } \frac{1}{k^2+3} < \frac{1}{N^2+3} \\
&= \frac{1}{\left(\sqrt{\frac{1}{\epsilon}}\right)^2+3} \text{ because } N = \sqrt{\frac{1}{\epsilon}} \\
&= \frac{1}{\left(\frac{1}{\epsilon}\right)+3} \\
&< \frac{1}{\left(\frac{1}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}$$

ANSWER (another possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \sqrt{\frac{1}{\epsilon} - 3}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{1}{k^2+3} - 0\right| \\
&= \frac{1}{k^2+3} \\
&< \frac{1}{N^2+3} \text{ since } k > N \text{ implies that } \frac{1}{k^2+3} < \frac{1}{N^2+3} \\
&= \frac{1}{\left(\sqrt{\frac{1}{\epsilon} - 3}\right)^2 + 3} \text{ because } N = \sqrt{\frac{1}{\epsilon} - 3} \\
&= \frac{1}{\left(\frac{1}{\epsilon} - 3\right) + 3} \\
&= \frac{1}{\left(\frac{1}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}$$

(b) $a_n = \frac{3n+2}{2n-1}$, $L = \frac{3}{2}$.

ANSWER (a possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \dfrac{7}{4\epsilon} + \dfrac{1}{2}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{3k+2}{2k-1} - \frac{3}{2}\right| \\
&= \left|\frac{3k+2-\frac{3}{2}(2k-1)}{2k-1}\right| \\
&= \left|\frac{3k+2-3k+\frac{3}{2}}{2k-1}\right| \\
&= \left|\frac{\frac{7}{2}}{2k-1}\right| \\
&= \frac{\frac{7}{2}}{2k-1} \\
&< \frac{\frac{7}{2}}{2N-1} \quad \text{because } k > N, \text{ so } \frac{1}{2k-1} < \frac{1}{2N-1} \\
&= \frac{\frac{7}{2}}{2\left(\frac{7}{4\epsilon}+\frac{1}{2}\right)-1} \quad \text{since } N = \frac{7}{4\epsilon}+\frac{1}{2} \\
&= \frac{\frac{7}{2}}{\left(\frac{7}{2\epsilon}+1\right)-1} \\
&= \frac{\frac{7}{2}}{\left(\frac{7}{2\epsilon}\right)} \\
&= \frac{7}{2}\cdot\frac{2\,\epsilon}{7} \\
&= \epsilon.
\end{aligned}$$

(c) $a_n = \frac{n^2+2}{n^2-3}$, $L = 1$.

ANSWER (a possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \sqrt{\frac{5}{\epsilon} + 3}$. Note that $N > 1$. Then, if $k > N$, we have

$$
\begin{aligned}
|a_k - L| &= \left| \frac{k^2+2}{k^2-3} - 1 \right| \\
&= \left| \frac{k^2+2-(k^2-3)}{k^2-3} \right| \\
&= \left| \frac{2+3}{k^2-3} \right| \\
&= \left| \frac{5}{k^2-3} \right| \\
&= \frac{5}{k^2-3} \quad \text{because } k > N \text{ and } N > 1, \text{ so } k \leq 2 \\
&< \frac{5}{N^2-3} \quad \text{since } k > N, \text{ so } \frac{1}{k^2-3} < \frac{1}{N^2-3} \\
&= \frac{5}{\left(\sqrt{\frac{5}{\epsilon}+3}\right)^2 - 3} \quad \text{because } N = \sqrt{\frac{1}{\epsilon}+3} \\
&= \frac{5}{\frac{5}{\epsilon}+3-3} \\
&= \frac{5}{\left(\frac{5}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}
$$

ANSWER (another possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \frac{5}{\epsilon} + 3$. Note that $N > 1$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{k^2+2}{k^2-3} - 1\right| \\
&= \left|\frac{k^2+2-(k^2-3)}{k^2-3}\right| \\
&= \left|\frac{2+3}{k^2-3}\right| \\
&= \left|\frac{5}{k^2-3}\right| \\
&= \frac{5}{k^2-3} \quad \text{because } k > N \text{ and } N > 1 \text{, so } k \leq 2 \\
&< \frac{5}{N^2-3} \quad \text{since } k > N \text{, so } \frac{1}{k^2-3} < \frac{1}{N^2-3} \\
&= \frac{5}{\left(\frac{5}{\epsilon}+3\right)^2 - 3} \quad \text{because } N = \frac{1}{\epsilon} + 3 \\
&= \frac{5}{\left(\frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 9\right) - 3} \quad \text{because } N = \frac{1}{\epsilon} + 3 \\
&= \frac{5}{\frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 6} \\
&< \frac{5}{\left(\frac{30}{\epsilon}\right)} \quad \text{because } \frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 6 \ > \ \frac{30}{\epsilon} \\
&= \frac{5}{\left(\frac{30}{\epsilon}\right)} \\
&= \frac{\epsilon}{6} \\
&< \epsilon.
\end{aligned}$$

7. (Key words: Sec 11.1, sequence, squeeze theorem) The sequence $a_n = (5n+4)/(3n^2-2)$ converges to 0. Justify this fact by squeezing a_n between 0 and another sequence of type $b_n = (\text{a constant})/n$ and using the Squeeze theorem. You may assume $\lim_{n\to\infty} (\text{any constant})/n = 0$.

   See a similar (model) solution at: `https://egunawan.github.io/fall17/hw/models.pdf`

Scratch work (do not submit) for Question 7: You can try some constant and check if that works, but the following is a sure way to get b_n which works.

Think, I want to find $A\ n$ bigger than the numerator $5n+4$ and $B\ n^2$ smaller than the denominator $3n^2-2$ for all $n \geq$ some number. I want:

$$5n+4 \leq A\ n \quad \text{(note how both sides are positive whenever } n \geq 1 \text{ and } B \text{ is positive)} \Leftrightarrow$$

$$\frac{5n+4}{n} \leq A \Leftrightarrow$$

$$5+\frac{4}{n} = \frac{5n}{n}+\frac{4}{n} \leq A.$$

Since the left hand-side of this inequality is a decreasing sequence (for $n = 1, 2, 3, \ldots$), we have $5+\frac{4}{n} \leq 5+\frac{1}{4}$, so I can choose $A = 5+\frac{1}{4}$ or $A = 6$.

I want:

$$B\ n^2 \leq 3n^2-2 \quad \text{(note how both sides are positive whenever } n \geq 1 \text{ and } B \text{ is positive)} \Leftrightarrow$$

$$B \leq \frac{3n^2-2}{n^2} \Leftrightarrow$$

$$B \leq \frac{3n^2}{n^2}-\frac{2}{n^2} \Leftrightarrow$$

$$B \leq 3-\frac{2}{n^2}.$$

Since the right hand-side of this inequality is an increasing sequence (for $n = 1, 2, 3, \ldots$), we have $3-\frac{2}{1^2} \leq 3-\frac{2}{n^2}$ (for $n \geq 1$), so I can choose $B = 3-2 = 1$ or any smaller positive B.

ANSWER (what I submit) for Question 7 (a possible answer): Note that, if $n = 1, 2, 3, \ldots$, we have

$$0 \leq \frac{5n+4}{3n^2-2},$$

and

$$\begin{aligned}\frac{5n+4}{3n^2-2} &\leq \frac{6n}{3n^2-2} \quad \text{because } 5n+4 \leq 6n \text{ for } n \geq 1\\ &\leq \frac{6n}{n^2} \quad \text{because } n \leq 3n^2-2 \text{ for } n \leq 1\\ &= \frac{6}{n}.\end{aligned}$$

Since

$$0 \leq \frac{5n+4}{3n^2-2} \leq \frac{6}{n} \quad \text{for } n \geq 1$$

AND

$$\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{6}{n},$$

by the Squeeze Theorem (Stewart top of pg 698 and see also Fig. 7) we have

$$\lim_{n\to\infty} \frac{5n+4}{3n^2-2} = 0.$$

8. (Key words: Sec 11.2 series, decimal expansion)

   (a) Use geometric series to write $2.74\bar{9} = 2.74999999\ldots$ as a fraction.

ANSWER:

$$
\begin{aligned}
2.74\bar{9} &= 2.74999999\ldots \\
&= 2.74 + 9\frac{1}{100} + 9\frac{1}{1000} + 9\frac{1}{10000} + \ldots \\
&= 2.74 + \frac{9}{100}\frac{1}{10} + \frac{9}{100}\frac{1}{100} + \frac{9}{100}\frac{1}{1000} + \ldots \\
&= 2.74 + \frac{9}{100}\left(\frac{1}{10}\right)^1 + \frac{9}{100}\left(\frac{1}{100}\right)^2 + \frac{9}{100}\left(\frac{1}{1000}\right)^3 + \ldots \\
&= 2.74 + \frac{9}{100}\sum_{k=2}^{\infty}\left(\frac{1}{10}\right)^{k-1} \\
&= 2.74 - \frac{9}{100}(\frac{1}{10})^0 + \frac{9}{100}\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^{k-1} \\
&= 2.74 - \frac{9}{100} + \frac{9}{100}\frac{1}{1-\frac{1}{10}} \\
&= 2.74 - \frac{9}{100} + \frac{9}{100}\frac{1}{(\frac{9}{10})} \\
&= \frac{274}{100} - \frac{9}{100} + \frac{10}{100} \\
&= \frac{275}{100}.
\end{aligned}
$$

(b) Write a *different* decimal expansion for 2.74999999.... (Hint: can you use your solution above?)

2.75 is the other decimal expansion for 2.74999999...

(c) i. Find the (*exact*, not rounded up) repeating decimal expansions of $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$, and $\frac{6}{7}$. You should practice doing at least two of them without the help of a computer (hint: do long division) so that you can do this on a test, but you don't need to include it in your submission.

ii. List them in a table and note the interesting pattern.

1/7	$0.\overline{142857}$ or $0.\mathbf{1}42857\overline{142857}$
2/7	$0.\overline{285714}$ or $0.\mathbf{2}857\overline{142857}$
3/7	$0.\overline{428571}$ or $0.\mathbf{4}2857\overline{142857}$
4/7	$0.\overline{571428}$ or $0.\mathbf{57}\overline{142857}$
5/7	$0.\overline{714285}$ or $0.\mathbf{7}\overline{142857}$
6/7	$0.\overline{857142}$ or $0.\mathbf{8}57\overline{142857}$

iii. Choose just ONE the six fractions above and show that your repeating decimal expansion is indeed the same as the fraction. If you don't enjoy working with many digits, you can use a computing tool - make sure it gives you exact fractions (for example, WolframAlpha would work).

ANSWER (a possible answer):

$$
\begin{aligned}
0.\overline{571428} &= 571428\left(\frac{1}{10^6}\right)^1 + 571428\left(\frac{1}{10^6}\right)^2 + 571428\left(\frac{1}{10^6}\right)^3 + \dots \\
&= 571428\sum_{k=2}^{\infty}\left(\frac{1}{10^6}\right)^{k-1} \\
&= -571428\left(\frac{1}{10^6}\right)^0 + 571428\sum_{k=1}^{\infty}\left(\frac{1}{10^6}\right)^{k-1} \\
&= -571428 + 571428\ \frac{1}{1-\frac{1}{10^6}} \\
&= -571428 + 571428\ \frac{10^6}{999,999} \\
&= -571428 + (4,000,000/7) \\
&= 4/7
\end{aligned}
$$

9. (Key words: Sec 11.2 series) Do just ONE of the following:

Option I: There is a famous set C which is formed by repeating the following process infinitely many times. Start with the interval $C_0 = [0,1] = \{x \text{ a real number} : 0 \le x \le 1\}$. Remove the open middle third of this interval. Now you are left with $C_1 = [0,1/3] \cup [2/3,1]$. Next, remove the open middle thirds of each of the remaining intervals, i.e. remove $(1/9, 2/9)$ and $(7/9, 8/9)$. You are left with $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$.

(a) Draw a separate number line representing each of C_0, C_1, C_2 and C_3. It will help you visualize what is going on if you draw the number line for C_1 below the number line for C_0 and so on.

(b) What are the lengths of C_0, C_1, C_2 and C_3?

(c) Find an expression for the length of C_n.

(d) What are the lengths of the C? (Hint: use geometric series)

(e) Is C empty? If not, list some numbers that belong to C. Explain.

ANSWER:

(a) This is the famous *Cantor set.* You can google it.

(b)

object	area
C_0	1
C_1	$1-\left(\frac{1}{3}\right)$
C_2	$1-\left(\frac{1}{3}\right)-2\left(\frac{1}{3}\,\frac{1}{3}\right)=1-\left(\frac{1}{3}\right)-2\left(\frac{1}{3}\right)^2$
C_3	$1-\left(\frac{1}{3}\right)-2\left(\frac{1}{3}\right)^2-2^2\left(\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}\right)=1-\left(\frac{1}{3}\right)-2\left(\frac{1}{3}\right)^2-2^2\left(\frac{1}{3}\right)^3$

(c)

$$\begin{aligned}\text{Area of } C_n &= 1-(2)^0\frac{1}{3}-(2)^1\left(\frac{1}{3}\right)^2-(2)^2\left(\frac{1}{3}\right)^3-\;\dots\;-(2)^{n-1}\left(\frac{1}{3}\right)^n \\ &= 1-\frac{1}{3}\sum_{k=1}^{n}\left(\frac{2}{3}\right)^{k-1} \\ &= 1-\frac{1}{3}\left(\frac{1+\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}}\right) \quad \text{(why this formula? See Example 2 on Sec 11.2 page 709)}\end{aligned}$$

(d) The length of C is

$$\begin{aligned}\lim_{n\to\infty}\ (\text{Length of } C_n) &= 1-\frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) \\ &= 1-\frac{1}{3}\left(\frac{1}{\left(\frac{1}{3}\right)}\right) \\ &= 1-1 \\ &= 0.\end{aligned}$$

(e) Even though the length of C is 0, the Cantor set is not empty. For example, the numbers 0, 1, 1/3, 2/3, etc are never removed. In fact, there are infinitely many numbers in the Cantor set.

Option II: A two-dimensional counterpart of the above famous set is a ‘carpet’ square S which is formed by repeating the following process infinitely many times. Start with a filled square of side 1, call it S_0. Remove the center square (of side 1/3) of the original square and now you are left with S_1. Then remove the center squares (of side 1/9) of the eight smaller remaining squares to get S_2, and so on.

(a) Sketch S_0, S_1, and S_2 on the $x-y$ coordinate plane. Sketching S_3 is optional (lots of missing squares) but may be helpful.

(b) What are the areas of S_0, S_1, S_2 and S_3?

(c) Find an expression for the area of S_n.

(d) What is the area of the S? (Hint: use geometric series)

(e) Is S empty? If not, list some points that belongs to S. Explain.

ANSWER:

(a) This is the famous *Sierpinski carpet.* You can google it.

(b)

object	area
S_0	1
S_1	$1-\left(\frac{1}{3}\right)^2$
S_2	$1-\left(\frac{1}{3}\right)^2-8\left(\frac{1}{3}\,\frac{1}{3}\right)^2=1-\left(\frac{1}{3}\right)^2-8\left(\frac{1}{3}\right)^4$
S_3	$1-\left(\frac{1}{3}\right)^2-8\left(\frac{1}{3}\right)^4-8^2\left(\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}\right)^2=1-\left(\frac{1}{3}\right)^2-8\left(\frac{1}{3}\right)^4-8^2\left(\frac{1}{3}\right)^6$

(c)

$$\begin{aligned}\text{Area of } S_n &= 1-(8)^0\frac{1}{9}-(8)^1\left(\frac{1}{9}\right)^2-(8)^2\left(\frac{1}{9}\right)^3-\cdots-(8)^{n-1}\left(\frac{1}{9}\right)^n\\ &= 1-\frac{1}{9}\sum_{k=1}^{n}\left(\frac{8}{9}\right)^{k-1}\\ &= 1-\frac{1}{9}\left(\frac{1+\left(\frac{8}{9}\right)^n}{1-\frac{8}{9}}\right) \quad \text{(why this formula? See Example 2 on Sec 11.2 page 709)}\end{aligned}$$

(d) The area of S is

$$\begin{aligned}\lim_{n\to\infty}\text{Area of } S_n &= 1-\frac{1}{9}\left(\frac{1}{1-\frac{8}{9}}\right)\\ &= 1-\frac{1}{9}\left(\frac{1}{\left(\frac{1}{9}\right)}\right)\\ &= 1-1\\ &= 0.\end{aligned}$$

(e) Even though the area of S is 0, this set is not empty. For example, the four sides of the original S_0 are never removed.

10. (Write your own problem) Review the topics of Sec 11.1 and Sec 11.2.

(a) Write a (non-vocabulary) problem that you think requires a decent understanding of one or more of the concepts/ skills to solve. (For ideas, you may look at Stewart's textbook or other textbooks or WebAssign, but don't just copy the problem). The problem's solution should require more than simply writing down a definition from the textbook.

(b) Explain briefly why you chose to highlight certain concepts (for example, maybe you think it's interesting, difficult, or has applications to science).

(c) Write a complete solution to your problem.

(d) Show your problem to your classmates to get their feedback. Write a brief, one-sentence comment about what other people thought. One or two questions submitted from students may be chosen for future test questions.