Math 1152Q, Fall 2017 — model solutions

Week 1 - August 28, 2017

Here are a few problems that are similar to those on the first two weeks' assignment and quizzes. These written solutions are in the style I'd like to suggest you write. Credit: T. Chumley, used with thanks.

Problem 1. Compute the limit

$$\lim_{n\to\infty} n^2 e^{-n}.$$

Solution 1. Observe that

$$\begin{aligned}\lim_{n\to\infty} n^2 e^{-n} &= \lim_{n\to\infty} \frac{n^2}{e^n}\\ &\stackrel{1}{=} \lim_{n\to\infty} \frac{2n}{e^n}\\ &\stackrel{1}{=} \lim_{n\to\infty} \frac{2}{e^n}\\ &= 0,\end{aligned}$$

where the equalities labeled 1 are justified by L'Hopital's rule.

Problem 2. Compute the limit

$$\lim_{n\to\infty} \ln(2n^2+1) - \ln(n^2+1).$$

Solution 2. Observe that

$$\begin{aligned}\lim_{n\to\infty} \ln(2n^2+1) - \ln(n^2+1) &\stackrel{1}{=} \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right)\\ &\stackrel{2}{=} \ln\left(\lim_{n\to\infty} \frac{2n^2+1}{n^2+1}\right)\\ &= \ln(2),\end{aligned}$$

where the equality labeled 1 is because $\ln(a) - \ln(b) = \ln(a/b)$ [Note: some justifications like this one are a judgment call. It would be ok if you just used the rule of logs without citing it. You should try to err on the side of clarity] and the equality labeled 2 is due to the fact that the natural log function is continous at 2 and a theorem about limits and continuous functions (see, for example, Thm 8 on Sec 2.5).

Problem 3. Compute the limit

$$\lim_{n\to\infty} \frac{(-1)^{n+1}n}{\sqrt{2n^3+\sqrt{n}}}.$$

Solution 3. Let

$$a_n = \frac{-n}{\sqrt{2n^3+\sqrt{n}}}, \quad b_n = \frac{(-1)^{n+1}n}{\sqrt{2n^3+\sqrt{n}}}, \quad c_n = \frac{n}{\sqrt{2n^3+\sqrt{n}}}.$$

Observe that

$$a_n \leq b_n \leq c_n \quad \text{for all } n \geq 1.$$

Moreover,

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} \frac{n}{\sqrt{2n^3+\sqrt{n}}} \cdot \frac{\frac{1}{n^{3/2}}}{\frac{1}{n^{3/2}}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n^{1/2}}}{\sqrt{2n^3+\sqrt{n}}\sqrt{\frac{1}{n^3}}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n^{1/2}}}{\sqrt{\frac{2n^3+\sqrt{n}}{n^3}}} \\
&= \frac{\lim_{n\to\infty} \frac{1}{n^{1/2}}}{\sqrt{\lim_{n\to\infty} \frac{2n^3+\sqrt{n}}{n^3}}} \\
&= \frac{0}{\sqrt{2}} \\
&= 0.
\end{aligned}$$

Note $\lim_{n\to\infty} a_n = -\lim_{n\to\infty} c_n = 0$. Therefore, by the squeeze theorem, $\lim_{n\to\infty} b_n = 0$.