1. True or False.

(a) If $-1 < \alpha < 1$, then $\lim_{n\to\infty} \alpha^n = 0$. **T** **F**
Justification:
Answer: True. Since $\lim_{n\to\infty} |\alpha|^n = 0$, by Squeeze theorem we have $\lim_{n\to\infty} \alpha^n = 0$. See Sec 11.1 Ex 11 pg 698.

(b) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. **T** **F**
Justification:
Answer: False. Counterexample: $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$.

(c) if $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n + b_n\}$ is divergent. **T** **F**
Justification:
Answer: False. Counterexample: $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, but $a_n + b_n = 0$.

(d) if $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n b_n\}$ is divergent. **T** **F**
Justification:
Answer: False. Counterexample: $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, but $a_n b_n = -1$.

(e) If $a_n > 0$ for $n = 1, 2, 3, \ldots$ and $\{a_n\}_{n=1}^{\infty}$ is decreasing, then $\sum a_n$ converges. **T** **F**
Justification:
Answer: True by the monotonic sequence theorem, since $\{a_n\}$ is decreasing and bounded by 0 and a_1.

(f) The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ converges. **T** **F**
Justification:
Answer: False by p-series test since $\sin 1 \le 1$.

(g) The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ diverges. **T** **F**
Justification:
Answer: True by p-series test since $\sin 1 \le 1$.

(h) The series $\sum_{n=1}^{\infty} n^{-\cos 0}$ diverges. **T** **F**
Justification: Answer: True by p-series test since $\cos 0 = 1$.

(i) If $a_n > 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges **T** **F**
Justification: Answer: True. Why?

(j) The ratio test can be used to determine whether $\sum \frac{1}{n^4}$ converges. **T** **F**
Justification: Answer: False. Show computation.

(k) The ratio test can be used to determine whether $\sum \frac{1}{n!}$ converges. **T** **F**
Justification: Answer: True. Show computation.

(l) If $a_n > 0$ and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$, then $\lim_{n\to\infty} a_n = 0$ **T** **F**
Justification:
Answer: True due to Ratio Test and the fact that the terms of a convergent series have limit zero.

(m) If $a_n > 0$ and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 0$, then $\lim_{n\to\infty} a_n = 0$ **T** **F**
Justification:
Answer: True due to Ratio Test and the fact that the terms of a convergent series have limit zero.

(n) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent. **T** **F**
Justification:
Answer: False. Counterexample: $\sum \frac{1}{n}$ is divergent.

(o) If $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then $\lim_{n\to\infty} a_n = 0$ **T** **F**
Justification:
Answer: False. Counterexample: let $a_n = n$. Then $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{n+1}{n} = 1$, but $\lim_{n\to\infty} a_n = \infty$.

(p) $0.99999\cdots = 1$ **T** **F**
Justification: Answer: True. Show computation by applying the geometric series.

(q) $0.9999\cdots \ne 1$ **T** **F**
Justification:
Answer: False because of the computation from above.

(r) 0.66 is close to $\frac{2}{3}$ but $0.66 \ne \frac{2}{3}$ **T** **F**
Justification:
Answer: True. Show the computation which shows that $\frac{2}{3} - \frac{66}{100} \ne 0$.

(s) $0.66666\ldots$ is close to $\frac{2}{3}$ but $0.66666\cdots \neq \frac{2}{3}$ **T** **F**
Justification:
Answer: False. The geometric series computation shows that $0.66666\cdots = \frac{2}{3}$

(t) If 200 terms are added to a convergent series, the new series is still convergent **T** **F**
Justification: Answer: True

(u) If 200 terms are removed from a divergent series, the new series is covergent **T** **F**
Justification: Answer: False.

2. (7.8 improper integral)

i. When is an integral improper?

ii. Circle all the *improper* integrals and evaluate. Answer: All are improper except the first integral.

$$\int_2^3 \sqrt{x-2}\ \text{dx}, \quad \int_0^1 \frac{27}{x^5}\ \text{dx}, \quad \int_{-1}^1 \frac{1}{x}\ \text{dx}, \quad \int_2^\infty \left(\frac{1}{e^5}\right)^x \text{dx}, \quad \int_2^\infty \frac{1}{x^2+8x-9}\ \text{dx}$$

$$\int_0^1 \frac{4}{x^5}dx \text{ ans:divergent}, \quad \int_0^1 \frac{4}{x^{0.5}}dx \text{ ans:8}, \quad \int_2^3 \frac{2}{\sqrt{3-x}}dx \text{ ans: 4}, \quad \int_4^8 \frac{4}{x\sqrt{x^2-16}}dx \text{ ans:}\pi/3, \quad \int_{-7}^7 \frac{1}{\sqrt{49-x^2}}dx. \text{ ans:}\pi$$

iii. Write 2 improper (definite) integrals (different from above) so that one is convergent and the other is divergent.

iv. Write 2 proper (definite) integrals that are different from above.

v. Write 2 indefinite integrals.

vi. Determine whether $\int_0^1 9x^2 \ln(x)\ \text{dx}$ converges or diverges. If it converges, evaluate it.
Answer: -1

3. (WebAssign 9.1 differential equations)

(a) For what values of k does the function $y = \cos(kt)$ satisfy the differential equation $4y'' = -9y$?
Answer: $k = -\frac{3}{2}$, $k = \frac{3}{2}$

(b) Circle all functions which are solutions to $4y'' = -9y$. (Possibly none or all).

1. $y = -\cos(\frac{3t}{2})$
Answer: Yes
2. $y = \cos(\frac{3t}{2}) + 1$
Answer: No
3. $y = \sin(\frac{3t}{2})$
Answer: Yes
4. $y = \sin(\frac{3t}{2}) + \cos(\frac{3t}{2})$
Answer: Yes

(c) True or false? Every member of the family of functions $y = \dfrac{4\ln(x) + C}{x}$ is a solution of the differential equation

$$x^2y' + xy = 4$$

Answer: True. Show this by substituting y and y' into the differential equation.

(d) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(1) = 2$.
Answer: $y = \frac{4\ln(x)+2}{x}$.

(e) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(2) = 1$.
Answer: $y = \frac{4\ln(x)+2-4\ln(2)}{x}$.

(f) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(3) = 1$.
Answer: $y = \frac{4\ln(x)+3-4\ln(3)}{x}$.

(g) What can you say about a solution of the differential equation $y' = -\frac{1}{2}y^2$ *just* by looking at the differential equation? Circle all possibilities.

1. The function y must be equal to 0 on any interval on which it is defined.
   Answer: no.
2. The function y must be strictly increasing on any interval on which it is defined.
   Answer: no.
3. The function y must be increasing (or equal to 0) on any interval on which it is defined.
   Answer: no.
4. The function y must be decreasing (or equal to 0) on any interval on which it is defined.
   Answer: correct.
5. The function y must be strictly decreasing on any interval on which it is defined.
   Answer: no.

(h) Verify that all members of the family $y = \frac{2}{x+C}$ are solutions of the differential equation $y' = -\frac{1}{2}y^2$.

(i) Write a solution of the differential equation $y' = -\frac{1}{2}y^2$ that is not a member of the family $y = \frac{2}{x+C}$.
Answer: $y = 0$

(j) Find a solution of the initial-value problem. $y' = -\frac{1}{2}y^2 \qquad y(0) = 0.1$
Answer: $\frac{2}{x+20}$

(k) Find a solution of the initial-value problem. $y' = -\frac{1}{4}y^2 \qquad y(0) = 0.2$
Answer: $\frac{4}{x+20}$

(l) Find a solution of the initial-value problem. $y' = -\frac{1}{3}y^2 \qquad y(0) = 0.5$
Answer: $\frac{3}{x+6}$

(m) Find a solution of the initial-value problem. $y' = -\frac{1}{6}y^2 \qquad y(0) = 0.5$
Answer: $\frac{6}{x+12}$

(n) A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.1P\left(1 - \frac{P}{4000}\right)$$

1. For what values of P is the population increasing?
   Answer: $(0, 4000)$. Explanation: You need $1 - P/4000 > 0$ and $P > 0$.
2. For what values of P is the population decreasing?
   Answer: $(4000, \infty)$. Explanation: You need $1 - P/4000 < 0$ and $P > 0$.
3. What are the equilibrium solutions?
   Answer: $P = 4000$ and $P = 0$. Explanation: You need $dP/dt = 0$.

(o) A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 8y^3 + 15y^2.$$

1. What are the constant solutions of the equation?
   Answer: $y = 0, y = 3$, and $y = 5$
2. Sketch the polynomial $t^4 - 8t^3 + 15t^2$. In particular, mark the x-intercepts.
3. For what values of y is y increasing?
   Answer: When y is in one of the intervals $(-\infty, 0), (0, 3), (5, \infty)$
4. For what values of y is y decreasing?
   Answer: When y is in the interval $(3, 5)$

4. (9.1 Worksheet)

(a) True or false? Every differential equation has a constant solution. (If true, explain. If false, give a counterexample.)
Answer: False. Counterexample: think of a function $g(y)$ which has no zeros. You can use $\frac{dy}{dx} = g(y)$ as a counterexample.

(b) Consider the differential equation $\frac{dy}{dt} = 1 - 2y$.

i. Find all constant solution/s.
   Answer: $y = 1/2$

ii. Which of the following is a family of solutions? You may need to circle more than one.

$$y(t) = 1 + Ke^{-2t} \qquad y(t) = -Ke^{-2t} \qquad y(t) = \frac{1}{2} + Ke^{-2t} \qquad y(t) = \frac{1}{2} - Ke^{-2t}$$

Answer: $y(t) = \frac{1}{2} + Ke^{-2t} \qquad y(t) = \frac{1}{2} - Ke^{-2t}$

5. (9.3 reading homework)

(a) Draw a rough sketch of a possible solution to the logistic differential equation $\frac{dP}{dt} = 5P\left(1 - \frac{P}{8}\right)$. **You do not need to solve this differential equation to draw a rough sketch.** Hint: Explained in https://www.khanacademy.org/math/ap-calculus-bc/bc-diff-equations/bc-logistic-models/e/logistic-differential-equation

6. (9.3 WebAssign)

(a) Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = -9$$

Answer: $y = -\sqrt{x^2 + 81}$

(b) Find the solution of the differential equation that satisfies the given initial condition.

$$xy' + y = y^2, \quad y(1) = -8$$

Answer: $y = \frac{8}{8-9x}$

(c) Consider the differential equation $(x^2 + 15)y' = xy$.

i. Find all constant solutions.
Answer: $y = 0$

ii. Find all solutions.
Answer: $y = K\sqrt{x^2 + 15}$

(d) The differential equation below models the temperature of a 86^o C cup of coffee in a 20^o C room, where it is known that the coffee cools at a rate of 1^o C per minute when its temperature is 70^o C. Solve the differential equation to find an expression for the temperature of the coffee at time t. (Let y be the temperature of the cup of coffee in ^oC, and let t be the time in minutes, with $t = 0$ corresponding to the time when the temperature was 86^o C.)

$$\frac{dy}{dt} = -\frac{1}{50}(y - 20)$$

Answer: $y = Ke^{-t/50} + 20$. After considering the initial condition, we see that the temperature of the coffee at the time is described by $y = 66e^{-t/50} + 20$.

(e) A tank contains 8000 L of brine with 14 kg of dissolved salt. Pure water enters the tank at a rate of 80 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

1. How much salt is in the tank after t minutes?
Answer: $y = 14e^{-t/100}$ kg

2. How much salt is in the tank after 20 minutes?
Answer: $14e^{-0.2}$ kg. (Around 11.5 kg). You don't need to approximate.

(f) Find the orthogonal trajectories of the family of curves $y^2 = 8kx^3$. Sketch these orthogonal trajectories.
Answer: $2x^2 + 3y^2 = C$, a certain family of ellipses.

7. (Sec 10.3 Week 15 Quiz)

(a) Sketch the polar equation $r = \frac{5}{2}$

(b) Sketch the polar equation $\theta = \frac{\pi}{4}$

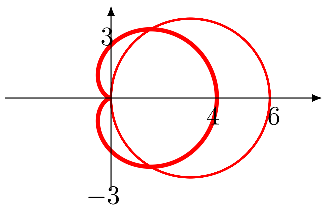
(c) Convert the polar equation $r = 3$ to Cartesian.
Answer: $x^2 + y^2 = 9$

(d) Convert the polar equation $\theta = \frac{\pi}{3}$.
Answer: $y = \sqrt{3}x$

(e) Convert the polar equation $\theta = \frac{\pi}{6}$.
Answer: $y = \frac{\sqrt{3}}{3}x$

(f) Convert the polar equation $r = 9\cos\theta$ to Cartesian.
Answer: $(x - 4.5)^2 + y^2 = (4.5)^2$

8. Consider the circle $r = 6\cos\theta$ and the cardioid $r = 2 + 2\cos\theta$.



(a) Mark points on *both* curves where $\theta = 0, \dfrac{\pi}{4}$, and $\dfrac{\pi}{2}$.

(b) Shade in the area inside the circle and outside the cardioid.

(c) Find the area (which you shade) inside the circle and outside the cardioid.
Answer: $\int_0^{\pi/3} 2(8\cos^2\theta - 1 - 2\cos\theta)d\theta = 4\pi$.

9. (Sec 10.1, 10.2 Week 12 quiz)

i. (a) Find *parametric* equations for the top half of the circle centered at $(2, 3)$ with radius 5, oriented *clockwise*.
Answer:

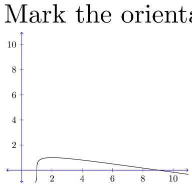
$$x = 2 + 5\cos(-t)$$
$$y = 3 + 5\sin(-t)$$
$$\text{for } \pi \le t \le 2\pi$$

(b) Eliminate the parameter to find a Cartesian equation of the curve.

ii. Consider the curve described by the parametric equations

$$x = t^3 + 1$$
$$y = 2t - t^2, \quad \text{for } -\infty < t < \infty$$

(a) Mark the orientation on the curve (direction of increasing values of t).



(b) Find the area enclosed by the x-axis and the given curve.
Answer:

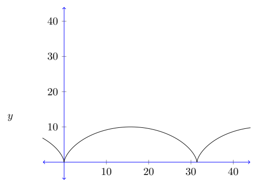
$$\begin{aligned}
\int_1^9 y \text{ dx} &= \int_{t=0}^{t=2} y(t)x'(t)dt \\
&= \int_0^2 (2t - t^2)(3t^2)dt \\
&= \int_0^2 (6t^3 - 3t^4)dt \\
&= \frac{6}{4}t^4 - \frac{3}{5}t^5\Big|_0^2 \\
&= \frac{3(16)}{2} - \frac{3(32)}{5} \\
&= \frac{3(40 - 32)}{5} \\
&= \boxed{\frac{24}{5}}
\end{aligned}$$

(c) Perform and describe a reality check by comparing your answer and the graph which has been drawn to scale.

iii. Consider the cycloid which is described by the parametric equations

$$x = 5(t - \sin t)$$
$$y = 5(1 - \cos t), \quad \text{for } \infty < t < \infty$$

(a) Mark the orientation on the curve (direction of increasing values of t).



(b) Find the area enclosed by the x-axis and *one* arch of the cycloid. Hint: $dx = 5(1 - \cos t)\ dt$.
Answer: $3\pi 5^2$

(c) Perform a reality check by comparing your answer and the graph (which is drawn to scale).

10. (Sec 11.10)

i. *If* f has a power series representation at 4, that is, if $f(x) = \sum_{n=0}^{\infty} c_n (x-4)^n$ for $|x - 4| < R$, then its coefficients are given by the formula $c_n =$ ________________. Answer: Theorem 5 on page 760.

Give a proof for your formula for c_n. Answer: follow pg 759 or lecture notes `https://egunawan.github.io/fall17/notes/notes11_10part1.pdf`.

ii. Circle all the true statements and cross out all the false statements, and justify.

(a) If the series $\sum_{n=1}^{\infty} c_n x^n$ converges for $|x| < R$, then $\lim_{n\to\infty} c_n x^n = 0$ for $|x| < R$.
Answer: True. See explanation, first sentence of pg 763. Or see Sec 11.2, Thm 6, pg 713.

(b) If the series $\sum_{n=1}^{\infty} c_n x^n$ diverges for $x = 5$, then $\lim_{n\to\infty} c_n x^n \neq 0$ for $x = 5$.
Answer: False. A counterexample: $c_n = \frac{1}{n\ 5^n}$. See Ex. 9 Sec 11.2, pg 713.

iii. Find the Maclaurin series for $f(x) = 6(1 - x)^{-2}$ using the definition of a Maclaurin series. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence.

Answer: Use Taylor series theorem/formulas 5,6,7 on pg 760. Follow Example 8 but replace $(1 + x)^k$ with $(1 - x)^{-2}$. The Maclaurin series is $\boxed{\sum_{n=0}^{\infty} 6(n+1)x^n}$.

Use Ratio Test to find the radius of convergence $\boxed{R = 1}$.

iv. Use a Maclaurin series given in this table `http://egunawan.github.io/fall17/quizzes/11_10_table01.pdf` (printed on the next page) to obtain the Maclaurin series for the function $f(x) = 8e^x + e^{8x}$. Find the radius of convergence.

Answer: Use the table to get $e^x = \sum_{n=1}^{\infty} x^n n!$. Apply the Composition Theorem with $h(x) = 8x$ and $f(t) = e^t$ to get $e^{8x} = \sum_{n=1}^{\infty} \frac{(8x)^n}{n!}$. Apply 'sum' theorem for series (pg 714 Sec 11.2) to get the sum $\boxed{\sum_{n=0}^{\infty} (8 + 8^n)\frac{x^n}{n!}. \text{ The series is convergent for all real numbers}}$.

v. Evaluate the indefinite integral $\left(8 \int \frac{e^x - 1}{5x} \text{dx}\right)$ as an infinite series.

Answer: Read Example 11 pg 768-769 for similar problem. This answer gives the Maclaurin series but you can choose a different Taylor series centered not at 0. First either use the table or directly evaluate the Maclaurin series for $e^x - 1 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$. Multiply this Maclaurin series by $\frac{1}{x}$ to get $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$. Apply term-by-term integration to get final answer, $\boxed{\frac{8}{5} \sum_{n=1}^{\infty} \frac{x^n}{(n)n!} + C}$.

vi. Find the Maclaurin series for $f(x) = e^{-4x}$ using the definition of a Maclaurin series. Don't use the table. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence R.

Answer: Follow Example 1 pg 760 but replace x with $-4x$. You get $\boxed{e^{-4x} = \sum_{n=0}^{\infty} \left(\frac{(-4)^n}{n!}\right) x^n. \text{ The series is convergent for all real numbers}}$.

11. (11.3 Integral test and p-series, 11.8 geometric series test/ratio test to find interval of convergence, 11.9 power series representation of a function, Week 8 quiz)

12. (Section 11.3) Suppose f is a continuous, positive, and decreasing function on $[1, \infty)$ and $a_k = f(k)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\ dx \qquad \sum_{k=1}^{5} a_k \qquad \sum_{k=2}^{6} a_k$$

13. (Sec 11.3 p-series) The *Riemann zeta*-function ζ ("zeta") is defined by

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

It is used in number theory to study the distribution of prime numbers.

(a) What is the domain of the function ζ? (That is, for what values of x is this function defined?)
Answer: Sec 11.3, page 722.

(b) Euler computed $\zeta(2)$ to be $\frac{\pi^2}{6}$. (See page 720, sec 11.3). Use this fact to find the sum of each series below.

$$\sum_{n=3}^{\infty} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(5n)^2} \qquad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

14. (Section 11.8 power series)

(a) What is a power series? (See Sec 11.8, top of page 747)

(b) In most cases, how do you find the radius of convergence of a power series?
Answer: to ratio test usually works. In certain situations, geometric series test works. See the Examples 1-5 in Sec 11.8, pg 747-750

(c) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$ Answer: see Example 5, pg 750.

(d) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} n!x^{2n}.$$ Answer: see Example 1, pg 747.

(e) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n^5}.$$ Answer: same R as Ex 2, pg 747, but both endpoints are included.

(f) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}.$$ Answer: same answer as Example 3, pg 748.

15. (Sec 11.9 WebAssign finding interval of convergence) For each function, find a power series representation and determine the interval of convergence.
(Check your work with WolframAlpha. Type "series representation of ...")

(a) $f(x) = \frac{1}{3+x}$ (see Sec 11.9 Example 2)

(b) $f(x) = \frac{x^3}{5+x}$ (see Sec 11.9 Example 3)

(c) $f(x) = \dfrac{x}{1+10x^2}$ (a variation of Sec 11.9 Example 3)

16. (Sec 11.9 WebAssign differentiation and integration of power series) For each function, find a power series representation. Determine the radius of convergence. (You do not need to determine the interval of convergence)

    (a) $f(x) = \dfrac{1}{(2+x)^2}$ (a variation of Sec 11.9 Ex 5)

    (b) $f(x) = \ln(1+x)$ (see Sec 11.9, Ex 6)

    (c) $f(x) = arctan(x)$ (see Sec 11.9, Ex 7)

    (d) $\displaystyle\int \frac{1}{1+x^7}$ dx (see Sec 11.9, Ex 8)

    (e) $\displaystyle\int \frac{x}{1-x^7}$ dx (a variation Sec 11.9, Ex 8)

17. (Ch 7 Integration methods) Decide whether the best method of integration is integration by parts, u-substitution, or trig substitution. Explain the first key step/s of evaluating the integrals (There often are more than one right answer).

    (a) $\displaystyle\int_0^4 \frac{\ln(x)}{\sqrt{x}}$ dx ______________________

    Answer: Integration by parts with $u = \ln(x)$ and $dv = \dfrac{1}{\sqrt{x}}$.

    (b) $\displaystyle\int \frac{1}{x\ln(x)}$ dx ______________________

    Answer: u-substitution for $\ln(x)$. Get $\boxed{\ln(\ln(x)) + \text{Constant}}$.

    (c) $\displaystyle\int_1^2 \ln(x)$ dx ______________________

    Answer: Integration by parts with (the only option) $u = \ln(x)$ and $dv =$ dx

    (d) $\displaystyle\int xe^{0.2x}$ dx ______________________

    Answer: Integration by parts with $u = x$ and $dv = e^{0.2x}$ dx

    (e) $\displaystyle\int_0^1 e^x \sin(x)$ dx ______________________

    Answer: Integration by parts with either $u = e^x$ and $dv = \sin(x)$ dx OR $dv = e^x$ dx and $u = \sin(x)$

    (f) $\displaystyle\int \frac{1}{x^2+2x+4}$ dx ______________________

    Answer: Complete the square, then use inverse trig substitution with $u = \sqrt{3}\,tan(\theta)$ or $u = \sqrt{3}\cot(\theta)$

18. Evaluate $\displaystyle\int \frac{1}{x^2+25}$ dx

    Answer: Do inverse trig substitution $x = 5\tan(\theta)$ so that you get $\displaystyle\int \frac{1}{x^2+25}\,\text{dx} = \int \frac{1}{5}\theta\, d\theta = \frac{1}{5}\arctan\left(\frac{x}{5}\right) + c.$

19. (Section 7.4 WebAssign partial fraction decomposition)

    (a) Evaluate $\displaystyle\int \frac{10}{(x+5)(x-2)}$ dx

    Answer: $\boxed{\frac{10(\ln(x-2)-\ln(5+x))}{7}}$

    (b) Evaluate $\displaystyle\int \frac{x+4}{x^2+2x+5}$ dx

    (c) Evaluate $\displaystyle\int \frac{7x^2-6x+16}{x^3+4x}$ dx

20. (7.8 improper integral, 11.3 integral test)

i. (a) Find the values of p for which the integral $\int_e^\infty \frac{6}{x(\ln x)^p}$ dx converges. Evaluate the integral for these values of p.

Answer: u-substitution. Check cases $p = 1$, $p < 1$, and $p > 1$. $\boxed{p > 1 \text{ diverges}}$

(b) Determine whether

$$\int_2^\infty \left(\frac{1}{e^5}\right)^x \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: $\boxed{1/(5\ 10^e)}$

(c) Determine whether

$$\int_2^\infty \frac{1}{x^2 + 8x - 9} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: You can use partial fraction decomposition. $\boxed{\ln(11)/10}$

(d) Determine whether $\int_0^1 4x^{-5}$ dx is convergent or divergent. If it is convergent, evaluate it.
Answer: divergent

(e) Determine whether

$$\int_0^1 \frac{4}{x^{0.5}} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: $\boxed{8}$

(f) Determine whether

$$\int_2^3 \frac{2}{\sqrt{3-x}} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: $\boxed{4}$

ii. (a) Evaluate the integral $\int_1^\infty \frac{3}{x^6}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{3}{n^6}$ is convergent or divergent.

Answer = 3/5

(b) Evaluate the integral

$$\int_1^\infty \frac{1}{(4x+2)^3} \text{ dx.}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{1}{(4n+2)^3}$ is convergent or divergent.
Answer = 1/288

(c) Evaluate the integral $\int_1^\infty \frac{1}{\sqrt{x+9}}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{1}{\sqrt{n+9}}$ is convergent or divergent.

Answer: divergent

(d) Evaluate the integral $\int_1^\infty x\, e^{-9x}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty ne^{-9n}$ is convergent or divergent.

Answer: $10/81e^9$

(e) Are the conditions for the Integral Test satisfied for the series $\sum_{1}^{\infty} (\cos\ n)^2 + \frac{1}{n}$ dx ?

Answer: No. Why?

(f) Are the conditions for the Integral Test satisfied for the series $\sum_{1}^{\infty} \frac{(\cos\ n)^2}{n}$ dx ?

Answer: No. Why?

21. (Week 4 quiz)

i. (a) Consider $\sum \frac{5^n}{n!}$ and $\sum \frac{n!}{n^n}$ (similar to Sec 11.6 Example 5 pg 741).

(b) Show your work for attempting to use the ratio test to this series to determine whether it converges or diverges.

Answer:

For $a_n = \frac{5^n}{n!}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} \\ &= \lim_{n\to\infty} \frac{5}{n+1} \\ &= \lim_{n\to\infty} \frac{\frac{5}{n}}{1+\frac{1}{n}} \\ &= \frac{\lim_{n\to\infty} \frac{5}{n}}{1+\lim_{n\to\infty} \frac{1}{n}} \\ &= 0.\end{aligned}$$

For $a_n = \frac{n!}{n^n}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \frac{n^n}{n!} \\ &= \lim_{n\to\infty} (n+1) \frac{n^n}{(n+1)^{(n+1)}} \\ &= \lim_{n\to\infty} (n+1) \frac{n^n}{(n+1)^n (n+1)} \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\ &= \frac{1}{e} \text{ as given at the top of the page}\end{aligned}$$

See also book's solution for a similar-looking series Sec 11.6 Example 5 pg 741.

(c) If the ratio test is conclusive, determine whether the series is convergent or divergent. Otherwise, state that the ratio test is inconclusive.

Answer:

The two series converge by the ratio test since $0 < 1$ and $\frac{1}{e} < 1$.

ii. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln\ n}{n}$ and $\sum_{n=4}^{\infty} \frac{1}{2^n - 9}$ (from Sec 11.4 Examples 2 and 3) converge

Answer:

iii. Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ and $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ (from Sec 11.4 Examples 1 and 4) converge.

Answer:

22. (11.2: geometric and harmonic series, 11.4: comparison test, 11.6: ratio tests) STRATEGY TIPS:

✓ The ratio test usually works when the term contains factorial like $(n+3)!$ or exponents like 7^n, $\dfrac{1}{7^n}$.

✗ The ratio test will *not* work with series with ONLY p-series-like terms, for example, $\sum \frac{n^2+4}{\sqrt{n^5-1}}$. Convince yourself.

✓✗ Only use one of the comparison tests are when the series looks like the geometric series $\sum r^n$, or the p-series $\sum \frac{1}{n^p}$.
You can check all the 'does [blank] converge' questions below with WolframAlpha.

Show whether each series $\sum a_n$ converges. For full credit you should give

- The series $\sum b_n$ and an explanation why $\sum b_n$ converges/diverges (if you use Comparison or Limit Comparison test)
- An inequality or limit computation
  - If using the Comparison Test, give an inequality of the form $a_n \leq b_n$ or $a_n \leq b_n$
  - If using the Limit Comparison Test, compute $\lim_{n\to\infty} a_n/b_n$
- A conclusion statement.

i. (Book examples) Pg 728-730: Sec 11.4 Ex 1,2,3,4; Pg 740-741 Sec 11.6 Ex 3, 5

ii. (a) $\sum_{n=2}^{\infty} \dfrac{n^3}{n^4+1}$
Answer: divergent. Let $b_n = 1/n$ and use LCT.

(b) $\sum_{n=1}^{\infty} \dfrac{6^n}{5^n-1}$
Answer: divergent. Use Divergence test.

(c) $\sum_{n=1}^{\infty} \dfrac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$
Answer: convergent. Let $b_n = 1/n^2$ and use LCT.

iii. (Sec 11.4)

(a) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{n+6^n}{n+2^n}$ converges or diverges.
Answer: ✓LCT attempt 1: You try LCT with $\sum(\frac{6}{2})^n$ and it works.
✓LCT attempt 2: LCT with $\sum \frac{1}{n}$ also works. But this may not be the first thing that comes to your mind.
✓Divergence test: the terms are increasing, so this test works.
✓Comparison test: find a big enough constant A so that $a_n > A(\frac{6}{2})^n$.
✓Ratio test: you see powers, so you try the ratio test. The ratio $\frac{a_{n+1}}{a_n}$ goes to 6/2.

(b) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{2n+3^n}{2n+7^n}$ converges or diverges.
Answer: ✓LCT attempt 1: You try LCT with $\sum(\frac{3}{7})^n$ and it works.
✓LCT attempt 2: LCT with $\sum \frac{1}{n^2}$ also works, but this may not be the first thing that comes to your mind.
✗Divergence test: inconclusive.
✓Comparison test: find a big enough constant A so that $a_n < A(\frac{3}{7})^n$.
✓Ratio test: you see powers, so you try the ratio test. The ratio $\frac{a_{n+1}}{a_n}$ goes to 3/7.

(c) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{5n^2-1}{6n^4+7}$ converges or diverges. (Hint: compare with a p-series and use one of the comparison tests. Would the ratio test be conclusive? See strategy above.)

(d) Determine whether the series $\sum_{n=6}^{\infty} \dfrac{n-5}{n7^n}$ converges or diverges. (Hint: compare with a geometric series. You can also try the ratio test because you see powers $(\frac{1}{7})^n$)

(e) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{5^n}{n7^n}$ converges or diverges. (Hint: compare with a geometric series. You can also try the ratio test because you see powers $(\frac{5}{7})^n$)

(f) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{5^{2n}}{n7^n}$ converges or diverges.
Answer:
✗LCT with geometric series $\sum(\frac{25}{7})^n$ is inconclusive.
✓LCT with comparing with the harmonic series $\sum \frac{1}{n}$ works.
✓You try the ratio test because you see powers $(\frac{25}{7})^n$.

(g) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{n+8}{n\sqrt{n}}$ converges or diverges.

Answer:

✓LCT: you compare with a p-series because it looks like one.

✗Ratio test: you try ratio test and it's inconclusive. Recall that the ratio test never works for any series that looks ONLY like a p-series.

iv. (Divergence Test Sec 11.2)

(a) True or false? If a_n does not converge to 0, then the series of $\sum_{n=1}^{\infty} a_n$ diverges.
Answer: True, by divergence test.

(b) True or false? If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
Answer: False. Counterexample: the harmonic series is divergent even though its terms converge to 0.

(c) Let $a_n = \frac{4n}{7n+1}$. (I) Determine whether $\{a_n\}$ is convergent.
Answer: The sequence $\{a_n\}$ converges to 4/7.
(II) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.
Answer: The series diverges by the divergence test.

(d) Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2-1}{100+5n^2}$ is convergent or divergent.
Answer: The series diverges by the divergence test.

v. (Sec 11.6)

(a) $\sum_{n=1} \dfrac{5n!}{2^n}$ (hint: see factorial, think ratio test)

(b) $\sum_{n=1} \dfrac{n}{5^n}$ and $\sum_{n=1} ne^{-5n}$

Answer: ✗LCT with geom. series $\sum(\frac{1}{5})^n$ is inconclusive.

✓LCT comparing with $\sum \frac{1}{n^2}$ works.

✓You see power $(\frac{1}{5})^n$, so you try ratio test.

(c) $\sum_{n=1} \left(\dfrac{1}{4n+1}\right)^n$

Answer:

✓LCT: compare with p-series like $\sum \frac{1}{n^2}$.

✓LCT: compare with geometric series like $\sum \frac{1}{4^n}$.

✓Can use ratio test because you see powers something^n, but the computation for the ratio test is long.

(d) $\sum_{n=1} n\left(\dfrac{5}{7}\right)^n$

Answer: The comparison tests with geometric series $\sum(\frac{5}{7})^n$ are inconclusive. Try ratio test because you see power $(\frac{5}{7})^n$. Or use a comparison test with the p-series $\sum \frac{1}{n^2}$.

(e) $\sum_{n=1} \dfrac{|\sin(5n)|}{5^n}$

Answer:

✓see sin and $(\frac{1}{5})^n$, so think comparison test with the geometric series $\sum(\frac{1}{5})^n$.

✗The LCT with $b_n = (\frac{1}{5})^n$ fails.

✓The LCT with $b_n = \frac{1}{n^2}$ works.

✗Ratio test fails.)

(f) $\sum_{n=1} \dfrac{|\sin(5n)|}{n^5}$

Answer: ✓see sin and $(\frac{1}{n^5})$, so think the (non-limit) comparison test with the p-series $\sum(\frac{1}{n^5})$.

✗The LCT with $b_n = (\frac{1}{n^5})$ fails.

✓The LCT with $\frac{1}{n^3}$ works.

✗Ratio test fails.

(g) $\sum_{n=1} \dfrac{2^n}{n^3}$.

Answer:

✓Divergence test: numerator grows faster than the denominator, so use divergence test.

✗LCT: you see 2^n, but find that the comparison tests with the geometric series $\sum 2^n$ are inconclusive.

✓LCT: you try LCT with $\sum \frac{1}{n}$ and find that it works.

✓Ratio test: you can try ratio test because you see power 2^n.

✓Root test (not required to memorize): you can try root test because you see power 2^n.

(h) $\sum_{n=1} \dfrac{n!}{100^n}$
Answer:
✓Ratio test: you see *factorial* and exponent 100^n, so think ratio test.
✓Divergence test: you remember than factorial grows faster than exponential.
✗LCT: You try comparing it with $\sum n!$ but the result is inconclusive.

(i) (no. 8) $\sum_{n=1} \dfrac{n}{\sqrt{n^3+4}}$
Answer: ✓LCT or comparison: looks like a p-series, so use either.
✗Ratio test: The ratio test is inconclusive. The ratio test will not work for any p-series-like series.

23. (Sec 11.1)

i. (a) Compute $\lim_{n\to\infty}\left(1+\dfrac{1}{2n}\right)^n$ and $\lim_{n\to\infty}\left(1+\dfrac{5}{4n}\right)^n$ if they exist. (Hint: Notice the indeterminate form of type "1^∞".)
Answer: $\sqrt{e}$ and $e^{\frac{5}{4}}$

(b) Determine whether the sequence $\left\{\dfrac{5n!}{2^n}\right\}_{n=1}^{\infty}$ converges or diverges.
Answer: diverges.

ii. Each of the following series a_n converges to a limit L. Given $\epsilon > 0$, find *a* positive number N such that, if $n > N$, then a_n is within distance ϵ of L. SEE EXAMPLE `https://egunawan.github.io/fall17/notes/notes11_1choosingN.pdf`

(a) $a_n = \frac{1}{n^2+3}$, $L = 0$.
Answer (a possible answer):

Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \sqrt{\frac{1}{\epsilon}}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{1}{k^2+3} - 0\right| \\
&= \frac{1}{k^2+3} \\
&< \frac{1}{N^2+3} && \text{since } k > N \text{ implies that } \frac{1}{k^2+3} < \frac{1}{N^2+3} \\
&= \frac{1}{\left(\sqrt{\frac{1}{\epsilon}}\right)^2 + 3} && \text{because } N = \sqrt{\frac{1}{\epsilon}} \\
&= \frac{1}{\left(\frac{1}{\epsilon}\right) + 3} \\
&< \frac{1}{\left(\frac{1}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}$$

ANSWER (another possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \sqrt{\frac{1}{\epsilon} - 3}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{1}{k^2+3} - 0\right| \\
&= \frac{1}{k^2+3} \\
&< \frac{1}{N^2+3} && \text{since } k > N \text{ implies that } \frac{1}{k^2+3} < \frac{1}{N^2+3} \\
&= \frac{1}{\left(\sqrt{\frac{1}{\epsilon} - 3}\right)^2 + 3} && \text{because } N = \sqrt{\frac{1}{\epsilon} - 3} \\
&= \frac{1}{\left(\frac{1}{\epsilon} - 3\right) + 3} \\
&= \frac{1}{\left(\frac{1}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}$$

(b) $a_n = \frac{3n+2}{2n-1}$, $L = \frac{3}{2}$.
Answer:

ANSWER (a possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \dfrac{7}{4\epsilon} + \dfrac{1}{2}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{3k+2}{2k-1} - \frac{3}{2}\right| \\
&= \left|\frac{3k+2-\frac{3}{2}(2k-1)}{2k-1}\right| \\
&= \left|\frac{3k+2-3k+\frac{3}{2}}{2k-1}\right| \\
&= \left|\frac{\frac{7}{2}}{2k-1}\right| \\
&= \frac{\frac{7}{2}}{2k-1} \\
&< \frac{\frac{7}{2}}{2N-1} \quad \text{because } k > N\text{, so } \frac{1}{2k-1} < \frac{1}{2N-1} \\
&= \frac{\frac{7}{2}}{2\left(\frac{7}{4\epsilon}+\frac{1}{2}\right)-1} \quad \text{since } N = \frac{7}{4\epsilon}+\frac{1}{2} \\
&= \frac{\frac{7}{2}}{\left(\frac{7}{2\epsilon}+1\right)-1} \\
&= \frac{\frac{7}{2}}{\left(\frac{7}{2\epsilon}\right)} \\
&= \frac{7}{2}\cdot\frac{2\,\epsilon}{7} \\
&= \epsilon.
\end{aligned}$$

(c) $a_n = \frac{n^2+2}{n^2-3}$, $L = 1$.
Answer:

ANSWER (a possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \sqrt{\dfrac{5}{\epsilon}+3}$. Note that $N > 1$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{k^2+2}{k^2-3} - 1\right| \\
&= \left|\frac{k^2+2-(k^2-3)}{k^2-3}\right| \\
&= \left|\frac{2+3}{k^2-3}\right| \\
&= \left|\frac{5}{k^2-3}\right| \\
&= \frac{5}{k^2-3} \quad \text{because } k > N \text{ and } N > 1\text{, so } k \leq 2 \\
&< \frac{5}{N^2-3} \quad \text{since } k > N\text{, so } \frac{1}{k^2-3} < \frac{1}{N^2-3} \\
&= \frac{5}{\left(\sqrt{\frac{5}{\epsilon}+3}\right)^2-3} \quad \text{because } N = \sqrt{\frac{1}{\epsilon}+3} \\
&= \frac{5}{\frac{5}{\epsilon}+3-3} \\
&= \frac{5}{\left(\frac{5}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}$$

ANSWER (another possible answer): Let ϵ be a positive number (for simplicity, assume ϵ is smaller than 1). I choose $N = \frac{5}{\epsilon} + 3$. Note that $N > 1$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left| \frac{k^2+2}{k^2-3} - 1 \right| \\
&= \left| \frac{k^2+2-(k^2-3)}{k^2-3} \right| \\
&= \left| \frac{2+3}{k^2-3} \right| \\
&= \left| \frac{5}{k^2-3} \right| \\
&= \frac{5}{k^2-3} \quad \text{because } k > N \text{ and } N > 1 \text{, so } k \le 2 \\
&< \frac{5}{N^2-3} \quad \text{since } k > N \text{, so } \frac{1}{k^2-3} < \frac{1}{N^2-3} \\
&= \frac{5}{\left(\frac{5}{\epsilon}+3\right)^2 - 3} \quad \text{because } N = \frac{1}{\epsilon} + 3 \\
&= \frac{5}{\left(\frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 9\right) - 3} \quad \text{because } N = \frac{1}{\epsilon} + 3 \\
&= \frac{5}{\frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 6} \\
&< \frac{5}{\left(\frac{30}{\epsilon}\right)} \quad \text{because } \frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 6 \ > \ \frac{30}{\epsilon} \\
&= \frac{5}{\left(\frac{30}{\epsilon}\right)} \\
&= \frac{\epsilon}{6} \\
&< \epsilon.
\end{aligned}$$

iii. The sequence $a_n = (5n+4)/(3n^2-2)$ converges to 0. Justify this fact by squeezing a_n between 0 and another sequence of type $b_n = $ (a constant)$/n$ and using the Squeeze theorem. You may assume $\lim_{n\to\infty}$ (any constant)$/n = 0$. See a similar (model) solution at: `https://egunawan.github.io/fall17/hw/models.pdf`

Answer:

Scratch work (do not include) for Question iii: You can try some constant and check if that works, but the following is a sure way to get b_n which works.

Think, I want to find $A\ n$ bigger than the numerator $5n+4$ and $B\ n^2$ smaller than the denominator $3n^2-2$ for all $n \geq$ some number. I want:

$$\begin{aligned}
5n + 4 &\le A\ n \quad \text{(note how both sides are positive whenever } n \ge 1 \text{ and } B \text{ is positive)} \Leftrightarrow \\
\frac{5n+4}{n} &\le A \Leftrightarrow \\
5 + \frac{4}{n} = \frac{5n}{n} + \frac{4}{n} &\le A.
\end{aligned}$$

Since the left hand-side of this inequality is a decreasing sequence (for $n = 1, 2, 3, \ldots$), we have $5 + \frac{4}{n} \le 5 + \frac{1}{4}$, so I can choose $A = 5 + \frac{1}{4}$ or $A = 6$.

I want:

$$\begin{aligned}
B\ n^2 &\le 3n^2 - 2 \quad \text{(note how both sides are positive whenever } n \ge 1 \text{ and } B \text{ is positive)} \Leftrightarrow \\
B &\le \frac{3n^2-2}{n^2} \Leftrightarrow \\
B &\le \frac{3n^2}{n^2} - \frac{2}{n^2} \Leftrightarrow \\
B &\le 3 - \frac{2}{n^2}.
\end{aligned}$$

Since the right hand-side of this inequality is an increasing sequence (for $n = 1, 2, 3, \ldots$), we have $3 - \frac{2}{1^2} \le 3 - \frac{2}{n^2}$ (for $n \ge 1$), so I can choose $B = 3 - 2 = 1$ or any smaller positive B.

ANSWER (what I submit) for Question iii (a possible answer): Note that, if $n = 1, 2, 3, \ldots$, we have

$$0 \leq \frac{5n+4}{3n^2-2},$$

and

$$\begin{aligned}\frac{5n+4}{3n^2-2} &\leq \frac{6n}{3n^2-2} \quad \text{because } 5n+4 \leq 6n \text{ for } n \geq 1 \\ &\leq \frac{6n}{n^2} \quad \text{because } n \leq 3n^2-2 \text{ for } n \leq 1 \\ &= \frac{6}{n}.\end{aligned}$$

Since

$$0 \leq \frac{5n+4}{3n^2-2} \leq \frac{6}{n} \quad \text{for } n \geq 1$$

AND

$$\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{6}{n},$$

by the Squeeze Theorem (Stewart top of pg 698 and see also Fig. 7) we have

$$\lim_{n\to\infty} \frac{5n+4}{3n^2-2} = 0.$$

iv. Either give an example satisfying the condition or explain why no such sequence exists.

(a) A monotonically decreasing sequence that converges to 10.
Answer: (possible answers) $\left\{10 + \frac{1}{n}\right\}_{n=1}^{\infty}$ or $\left\{10 + \frac{1}{2^n}\right\}_{n=1}^{\infty}$

(b) A monotonically increasing bounded sequence that does not converge (that is, diverges).
Answer: No such sequence exists because the monotone sequence theorem says: *if a sequence is monotonically increasing and bounded, then it converges.*

(c) A non-monotonic sequence that converges to 4.
Answer: (a possible answer) $\{\left(\frac{-1}{2}\right)^n + \frac{3}{4}\}$ converges to $0 + \frac{3}{4}$ and is not monotonic.

24. (Sec 11.2 telescoping sum)

(a) (i.) Find a formula for the n-th partial sum S_n of the series $\sum_{k=2}^{\infty} \frac{5}{k^2-1}$.

(ii.) Evaluate $\lim_{n\to\infty} S_n$.

(iii.) Use the previous part to determine the sum of the series $\sum_{k=2}^{\infty} \frac{5}{k^2-1}$ or state that the series diverges.

(b) (i.) Find a formula for the n-th partial sum S_n of the series $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+3}}$.

(ii.) Evaluate $\lim_{n\to\infty} S_n$.

(iii.) Use the previous part to determine the sum of the series $\sum_{k=1}^{\infty} \frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+3}}$ or state that the series diverges.

25. (Sec 11.2 geometric series and decimal expansion)

(a) Use geometric series to write $2.74\bar{9} = 2.74999999\ldots$ as a fraction.
Answer:

$$
\begin{aligned}
2.74\bar{9} &= 2.74999999\ldots \\
&= 2.74 + 9\frac{1}{100} + 9\frac{1}{1000} + 9\frac{1}{10000} + \ldots \\
&= 2.74 + \frac{9}{100}\frac{1}{10} + \frac{9}{100}\frac{1}{100} + \frac{9}{100}\frac{1}{1000} + \ldots \\
&= 2.74 + \frac{9}{100}\left(\frac{1}{10}\right)^1 + \frac{9}{100}\left(\frac{1}{100}\right)^2 + \frac{9}{100}\left(\frac{1}{1000}\right)^3 + \ldots \\
&= 2.74 + \frac{9}{100}\sum_{k=2}^{\infty}\left(\frac{1}{10}\right)^{k-1} \\
&= 2.74 - \frac{9}{100}(\frac{1}{10})^0 + \frac{9}{100}\sum_{k=1}^{\infty}\left(\frac{1}{10}\right)^{k-1} \\
&= 2.74 - \frac{9}{100} + \frac{9}{100}\frac{1}{1-\frac{1}{10}} \\
&= 2.74 - \frac{9}{100} + \frac{9}{100}\frac{1}{(\frac{9}{10})} \\
&= \frac{274}{100} - \frac{9}{100} + \frac{10}{100} \\
&= \frac{275}{100}.
\end{aligned}
$$

(b) Write a *different* decimal expansion for $2.74999999\ldots$.
Answer: 2.75 is the other decimal expansion for $2.74999999\ldots$

(c) Write the repeating decimal expansion $0.\overline{571428}$ as a fraction.
Answer:

$$
\begin{aligned}
0.\overline{571428} &= 571428\left(\frac{1}{10^6}\right)^1 + 571428\left(\frac{1}{10^6}\right)^2 + 571428\left(\frac{1}{10^6}\right)^3 + \ldots \\
&= 571428\sum_{k=2}^{\infty}\left(\frac{1}{10^6}\right)^{k-1} \\
&= -571428\left(\frac{1}{10^6}\right)^0 + 571428\sum_{k=1}^{\infty}\left(\frac{1}{10^6}\right)^{k-1} \\
&= -571428 + 571428\,\frac{1}{1-\frac{1}{10^6}} \quad \text{(you should stop here \ldots)} \\
&= -571428 + 571428\,\frac{10^6}{999,999} \quad \text{(or here, since you don't have a computing tool)} \\
&= -571428 + (4,000,000/7) \\
&= 4/7
\end{aligned}
$$

(d) (i) Determine whether the series $\sum_{k=1}^{\infty} 2^{2k}3^{1-k}$ is convergent or divergent. (ii) If it is convergent, compute the sum.
Answer: a divergent geometric series

(e) (i) Determine whether the series

$$\sum_{k=1}^{\infty} 2^{2k}3^{5-2k} + 7^{k+1}10^{-k} + 3^{k+1}4^{-k}$$

is convergent or divergent. (ii) If it is convergent, compute the sum.
Answer: The sum of three convergent geometric series.

(f) (i) Find a closed-form formula for the nth term in the sequence $\left\{3, -4, \frac{16}{3}, -\frac{64}{9}, \ldots\right\}$. (ii) Use it to determine whether the series

$$3 - 4 + \frac{16}{3} - \frac{64}{9} + \ldots$$

is convergent or divergent. (iii) If it is convergent, compute the sum.
Answer: a divergent geometric series.

(g) (i) Determine whether the series $\sum_{k=1}^{\infty} \frac{(-1)^n}{e^{2n-1}}$ is convergent or divergent. (ii) If it is convergent, compute the sum.