Math 1152, Fall 2017 — Final Exam Fact Sheet

December 15

Definition of limit. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding positive N such that

$$\text{if} \quad n > N \quad \text{then} \quad |a_n - L| < \epsilon$$

Definition of limit at infinity. We write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty \text{ as } n \to \infty$$

if for every $M > 0$ there is a corresponding positive N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

L'Hôpital's rule. Suppose f, g are differentiable functions and $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are both 0 or both $\pm\infty$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Squeeze theorem for sequences.
If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Monotonic sequence theorem. Every bounded, monotonic sequence is convergent.

Definition of convergence for series. Consider the series $\sum_{n=1}^{\infty} a_n$. Then

$$s_N = \sum_{n=1}^{N} a_n$$

is called the Nth partial sum of the infinite series.

- if $\lim_{N\to\infty} s_N = s$ then the series $\sum_{n=1}^{\infty} a_n$ **converges** to s.
- if $\lim_{N\to\infty} s_N$ does not exist, then $\sum_{n=1}^{\infty} a_n$ **diverges**.

Geometric series. A series of the form $\sum ar^n$ is called a **geometric series**.

- if $|r| < 1$ then the series converges and $\sum_{n=0}^{\infty} ar^n = a/(1-r)$.
- if $|r| \geq 1$ then the series diverges.

Divergence test. Consider the series $\sum a_n$. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ **diverges**.

p-series test. A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is called a } p\text{-series.}$$

- if $p \leq 1$ then the series diverges
- if $p > 1$ then the series converges

Comparison test. Consider the series $\sum a_n$ with $a_n \geq 0$ for all n.

- if $a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges
- if $a_n \geq b_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ diverges

Limit comparison test. Consider the series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0.$$

Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Alternating series test. A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

where $b_n \geq 0$ for all n is called an alternating series. If

1. $b_{n+1} \leq b_n$ for all n large enough (ie. $\{b_n\}$ is an eventually decreasing sequence)
2. $\lim_{n\to\infty} b_n = 0$

then the series **converges**.

Ratio test. Consider the series $\sum_{n=1}^{\infty} a_n$ and suppose that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

1. if $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. if $L > 1$ or $L = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges.
3. if $L = 1$ then the test is inconclusive.

Fundamental Theorem of Calculus, part I. If f is continuous on $[a, b]$, then function g defined as

$$g(x) = \int_a^x f(t)\, dt, \quad a \leq x \leq b$$

satisfies $g'(x) = f(x)$.

Fundamental Theorem of Calculus, part II. If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any anti-derivative of f (ie. F is any function such that $F' = f$).

Integration by parts fomula. $\int u\, dv = uv - \int v\, du$

Integral test. If f is continuous, non-negative, and decreasing on $[1, \infty)$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(x)\, dx \text{ converges.}$$

Useful trig facts.

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2},$$

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Some derivatives. $\frac{d}{dx}b^x = \ln(b)b^x$

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = (\sec(x))^2$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\cot(x) = -(\csc(x))^2$$

Power series coefficients. If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Table 1: Important Maclaurin Series and their Radii of Convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Tangents and areas. Suppose f and g are differentiable functions. Consider the curve defined by the parametric equations

$$x = f(t)$$
$$y = g(t),$$

where y is a differentiable function of x. Then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \qquad \text{if} \qquad \frac{dx}{dt} \neq 0.$$

The area under the curve from $x = a$ to $x = b$ which is traced out *once* by the curve, $\alpha \leq t \leq \beta$, can be calculated as follows:

$$\int_a^b \text{y dx} = \int_\alpha^\beta g(t)\ f'(t)\ \text{dt, or}$$
$$\int_a^b \text{y dx} = \int_\beta^\alpha g(t)\ f'(t)\ \text{dt.}$$