1. Let $f(x)$ be continuous everywhere.

   (a) $\int_a^{\infty} f(x)\ \mathrm{dx} = \lim_{t\to\infty} \int_a^t f(x)\ \mathrm{dx}$ **T** **F**

   Justification:
   Answer: True, by definition

   (b) $\int_a^{\infty} f(x)\ \mathrm{dx} = \lim_{a\to\infty} \int_a^t f(x)\ \mathrm{dx}$ **T** **F**

   Justification:
   Answer: False, by definition

   (c) True of false? $\int_0^a f(x) = \int_0^a f(a-x)\ \mathrm{dx}$. If T, justify. If F, give a counterexample.
   Answer: True. (Hint: use u-substitution $u = a - x$.)

   (d) True of false? $\int_0^a f(x) = \int_0^a f(x-a)\ \mathrm{dx}$. If T, justify. If F, give a counterexample.
   Answer: False in general. Counterexample, let $f(x) = sin(x)$. Then $\int_0^{\pi} \sin(x)\ \mathrm{dx} = 2$ but $\int_0^{\pi} \sin(x - \pi)\ \mathrm{dx} = -2$. (Hint: use u-substitution $u = x - a$.)

   (e) Write a sanity-check-type calculation (different from what you've written above) to further confirm your answer in the previous two questions.

2. Evaluate the integral if possible. If they diverge, state that it diverges.

   (a)
   $$\int_0^1 \frac{27}{x^5}\ \mathrm{dx}$$
   Answer: $\boxed{\text{The integral diverges}}$.

   (b)
   $$\int_{-1}^1 \frac{1}{x}\ \mathrm{dx}$$
   Answer: $\boxed{\text{The integral diverges}}$.

3. **Multiple choice**

   i.) If $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} (xf(x))$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   Answer: $\boxed{\text{Not enough information}}$.

   ii.) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   Answer: $\boxed{\text{Not enough information}}$.

   iii.) If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   Answer: $\boxed{\text{zero}}$.

   iv.) If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0} (f(x) + g(x))$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   Answer: $\boxed{\infty}$

v.) If $\lim_{x\to\infty} f(x) = 1$, then $\lim_{x\to\infty} (f(x))^x$ is . . .
(a) zero (b) ∞ (c) 1 (d) another method is needed to determine this

Answer: not enough information

4. (not in Exam 2) Suppose that $f(1) = 3, f(4) = 8$, $f'(1) = 5$, $f'(4) = 5$ and f'' is continuous. Find the value of
$$\int_1^4 x f''(x)\ \mathrm{dx}.$$

5. (from 7.4 WebAssign, also in Week 7 sample quiz)
   (a) Decompose the following into partial fractions:
   $$\frac{3x^2 + 2x - 3}{x^3 - x}$$
   (b) Provide a computation that is either a formal verification (that is, a proof) or simply a sanity-check for your answer to the previous question.
   (c) Evaluate $\displaystyle\int \frac{10}{(x+5)(x-2)}\ \mathrm{dx}$.
   Answer: $10(\ln(x-2) - \ln(5+x))/7$
   (d) Provide a computation that is either a formal verification (that is, a proof) or simply a sanity-check for your answer to the previous question.

6. For the following questions, circle TRUE or FALSE. Justify briefly. (Hint: Section 11.5)
   (a) An application of the Alternating Series Estimation Theorem is the proof that e is irrational. **T** **F**

   Answer: True. Proof from class, Week 8 Monday.
   (b) An application of the Alternating Series Estimation Theorem is the proof that e is rational. **T** **F**

   Answer: False. The number e is not rational.
   (c) An application of the Alternating Series Estimation Theorem is a way to ensure that we can get an approximation to the *definite* integral $\displaystyle\int_0^1 e^{-x^2}\ \mathrm{dx}$ using series so that the approximation is within a certain error bound (for example 1/1000). **T** **F**

   Answer: True. See Example 11 Sec 11.10 pg 769.
   (d) We can always use the Alternating Series Estimation Theorem to ensure that we can get an approximation of a *function* using its Taylor polynomial so that the approximation is within a certain error bound (for example, 1/1000) on a certain interval. **T** **F**

   Answer: False. This only works when the resulting series (for every x in the domain of the function) is alternating. See Example 1 Sec 11.11 pg 775.
   (e) Suppose $b_k > 0$ for all k and $\displaystyle\sum_{k=1}^{\infty}(-1)^k\ b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \leq b_6$. **T** **F**

   Answer: True, by the Alternating Series Estimate Theorem
   (f) Suppose $b_k > 0$ for all n and $\displaystyle\sum_{k=1}^{\infty}(-1)^k\ b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \geq b_6$. **T** **F**

   Answer: False. The Alternating Series Estimate Theorem states that the inequality should go the other way.

7. (From WebAssign 11.8 no. 4)

   (a) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is 16. What is the radius of convergence of the power series $\sum c_n x^{4n}$?

   Answer: $\boxed{\sqrt[4]{16} = 2}$

   (b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{5n}$?

   Answer: $\boxed{\sqrt[5]{R}}$

   (c) The integral $\int_2^3 \sqrt{x-2}$ dx is improper. **T** **F**

   Answer: $\boxed{False}$

   (d) When is an integral improper? Hint: There are two kinds. See Sec 7.8 pg 527 and 531.

   (e) Write three definite improper integrals.

   (f) When is an integral proper? (The answer is when it is not improper, but explain what needs to happen for a definite integral to be proper).

   (g) Write three definite proper integrals.

8. (a) If the interval of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\sum_{n=1}^{\infty} n\, c_n x^{n-1}$? Why?

   Answer: $\boxed{R = 10}$ by Theorem 'term-by-term differentiation' Sec 11.9.

   (b) If the interval of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Why?

   Answer: $\boxed{R = 10}$ by Theorem 'term-by-term integration' Sec 11.9.

   (c) Use Table 1 to show that $\frac{d}{dx}\cos(x) = -\sin(x)$.

   (d) Write the first three nonzero terms of the Maclaurin series for $\tan(x)$ using Table 1 and long division of power series.
   Answer: Example 13 Sec 11.10 page 770.

   (e) Use the series that you just computed for $\tan(x)$ to evaluate

   $$\lim_{x \to 0} \frac{\tan(x) - x}{x^3}.$$

   Answer: $\boxed{1/3}$. See Week 11 Monday presentation.

   (f) Use a different method to evaluate

   $$\lim_{x \to 0} \frac{\tan(x) - x}{x^3}.$$

   Answer: $\boxed{1/3}$. See Week 11 Monday presentation.

   (g) Write the first three nonzero terms of the Maclaurin series for $e^x \sin(x)$ using Table 1 and multiplication of power series.
   Answer: Follow Example 13 Sec 11.10 page 770.

(h) Write the first three nonzero terms of the Maclaurin series for $\sec(x)$ using long division of power series and Table 1.
Answer: $\boxed{1 + x^2/2 + 4x^4/24}$

9. (from quiz week 8)

(a) What is a power series?
Answer: (See Sec 11.8, top of page 747)

(b) In most cases, how do you find the radius of convergence of a power series?
Answer: (See the test used in Examples 1-5 in Sec 11.8, pg 747-750)

(c) Find the *interval* of convergence of the following series.

- $\sum_{n=0}^{\infty} \frac{n\,(x-5)^n}{3^{n+1}}$. Answer: see Example 5, pg 750.
- $\sum_{n=0}^{\infty} n!\,(x-5)^n$. Answer: see Example 1, pg 747.
- $\sum_{n=0}^{\infty} \frac{(x-5)^{\frac{n}{2}}}{n}$. Answer: same radius of convergence as Example 2, pg 747.
- $\sum_{n=0}^{\infty} \frac{(x-5)^n}{n!}$. Answer: same answer as Example 3, pg 748.

10. (from quiz week 8) (i) Without using the table, find the power series representations of $\ln(1+x)$, $\arctan(x)$, and $\int \frac{x}{1-x^5}$ dx. Use the table to verify your answer.
(ii) Determine the *radius* of convergence and *interval* of convergence.
(iii) Write the first 5 non-zero terms of the resulting power series.

11. (from Sec 11.10 homework) Find the Taylor series for $f(x) = \sqrt{x}$ centered at 9.
Answer: Follow `https://socratic.org/questions/how-do-i-construct-the-taylor-series-for-f-x-sqrt-x-at-a` to get the coefficients for $n = 2, 3, 4, \ldots$

$$\begin{aligned} c_n &= \frac{1}{n!} f^{(n)}(9) \\ &= \frac{1}{n!}(-1)^{n-1}\frac{1}{2}\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\left(n-\frac{7}{2}\right)\cdots\frac{1}{2}(9)^{-(n-\frac{1}{2})} \\ &= \boxed{\frac{1}{n!}(-1)^{n-1}\frac{1}{2}\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\left(n-\frac{7}{2}\right)\cdots\frac{1}{2}\frac{3}{9^n}} \end{aligned}$$

The power series for $\sqrt{x}$ centered at $a = 9$ is

$$\boxed{3 + \frac{1}{6}(x-9) + \sum_{n=2}^{\infty} c_n (x-9)^n \text{ where } c_n \text{ is written is the boxed value above}}$$

You may wonder, how did the author in above link find that formula for $f^{(n)}(x)$?
After computing the first few terms for $n = 1, 2, 3, \ldots$:

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{2}x^{\frac{1}{2}-1} && = \frac{1}{2}x^{-(1-\frac{1}{2})} \\ f^{(2)}(x) &= -\left(1-\frac{1}{2}\right)\frac{1}{2}x^{\frac{1}{2}-1-1} && = -\left(1-\frac{1}{2}\right)\frac{1}{2}x^{-(2-\frac{1}{2})} \\ f^{(3)}(x) &= \left(2-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\frac{1}{2}x^{\frac{1}{2}-1-1-1} && = \left(2-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\frac{1}{2}x^{-(3-\frac{1}{2})}, \end{aligned}$$

the author started to notice a pattern (based on derivative rules for x^{constant}), and conclude that, for $n = 2, 3, 4, \ldots$.

$$\begin{aligned} f^{(n)}(x) &= (-1)^{n-1}\left((n-1) - \frac{1}{2}\right)\left((n-1) - 1 - \frac{1}{2}\right)\ldots\left(1 - \frac{1}{2}\right)\frac{1}{2}x^{-(n-\frac{1}{2})} \\ &= (-1)^{n-1}\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right)\ldots\left(1 - \frac{1}{2}\right)\frac{1}{2}x^{-(n-\frac{1}{2})} \end{aligned}$$

12. (From In-class worksheet on Week 9 Friday `https://egunawan.github.io/fall17/notes/1132q11_10_taylor_and_maclaurin_series.pdf`)
   (a) True or False? If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \ldots$ for $|x| < 1$ then $f'''(0) = 30$.
   Answer: True, since the coefficient for x^n is equal to $\dfrac{f(n)(0)}{n!}$ by definition of Maclaurin series. So $5 = \dfrac{f'''(0)}{3!}$, so $f'''(0) = 30$.
   (b) Can you write a Maclaurin series for $f(x) = \sqrt[3]{x}$? Explain why or why not.
   Answer: The function $f(x) = \sqrt[3]{x}$ is not differentiable at 0, so we cannot define a Maclaurin series for $f(x)$.

13. Compute a power series representation for $\ln(1 - x)$ using at least two different ways. (Check your answer with a technology).
   (a) (Way 1:)
   (b) (Way 2:)
   (c) Show at least two ways you can check your answer to this problem (without technology). The check could be a full verification or just a sanity check (not a full verification).
      1. Check 1
      2. Check 2

14. Use Maclaurin series to write the binomial series expansion of $(1 + x)^6$.
   Use Maclaurin series to write the binomial series expansion of $(1 + x)^k$ where k is a positive integer.

15. Using Table 1 (series), prove that

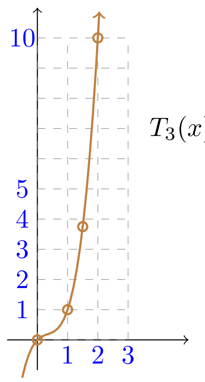
$$Re^{i\theta} = R\cos\theta + iR\sin\theta$$

16. True or False.
   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ **T** **F**
   Justification:
   Answer: True
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = -e$ **T** **F**
   Justification:
   Answer: False
   (c) If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \ldots$ converges for all x, then $f'''(0) = 2$ **T** **F**
   Justification:
   Answer: True, by definition of Maclaurin series.
   (d) If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \ldots$ converges for all x, then $f'''(0) = \frac{1}{2}$ **T** **F**
   Justification:
   Answer: False, by definition of Maclaurin series. See above answer.

17. (similar to WebAssign Sec 11.11)

(a) Find the 3rd degree Taylor polynomial of the function $f(x) = xe^{-2x}$ centered at 0. Sketch this polynomial. Label all the important points - label at least four convenient points.

Answer: $\boxed{T_3(x) = 2x^3 - 2x^2 + x}$



(b) (i) Approximate $f(x) = \ln(1 + 2x)$ by a Taylor polynomial of degree 3 centered at 1.

Answer: $\boxed{\ln 3 + 2/3(x-1) - 2/9(x-1)^2 + 8/81(x-1)^3}$.

(ii) Use Taylor's Inequality to estimate the accurate of the approximation $T_3(x)$ of $f(x)$ when $0.8 \leq x \leq 1.2$. You do not need to simplify your answer.

Answer:

Note that $f^{(4)}(x) = -96/(1 + 2x)^4$

$|R_3(x)| \leq \frac{M}{4!}|x-1|^4$, where $|f^{(4)}(x)| \leq M$. $0.8 \leq x \leq 1.2$ implies $-0.2 \leq x - 1 \leq 0.2$ which implies $|x - 1| \leq 0.2$ which implies $|x - 1|^4 \leq 0.2^4 = 0.0016$. The largest possible value for $|f^{(4)}(x)| = 96/(1 + 2x)^4$ in the interval is when $x = 0.8$, so we let $\boxed{M = |f^{(4)}(0.8)| = 96/(2.6)^4}$. So the error is within $\boxed{\frac{M}{4!}0.0016}$. You don't need to simplify M.

(c) Approximate $f(x) = e^{4x^2}$ by a Taylor polynomial with degree 3 centered at 0.

Answer: $\boxed{1 + 0x + 4x^2 + 0x^3 = 1 + 4x^2}$

18. (Integral Test from 11.3 WebAssign, also in Week 7 sample quiz)

(a) What are the conditions needed to apply the Integral Test ?

(b) Find the values of p for which the integral $\int_e^\infty \frac{6}{x(\ln x)^p}$ dx converges. Evaluate the integral for these values of p.

(Hint: Check what happens when $p = 1$, when $p < 1$, and when $p > 1$.)

Answer: $\boxed{p > 1 \text{ converges. Otherwise, diverges.}}$

(c) Evaluate the integral

$$\int_1^\infty \frac{3}{x^6} \text{ dx.}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series

$$\sum_1^\infty \frac{3}{n^6}$$

is convergent or divergent.

Answer = $\boxed{3/5}$

(d) Evaluate the integral

$$\int_1^\infty \frac{1}{(4x+2)^3} \text{ dx.}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series

$$\sum_{1}^{\infty} \frac{1}{(4n+2)^3}$$

is convergent or divergent.
Answer = $\boxed{1/288}$

(e) Evaluate the integral

$$\int_{1}^{\infty} \frac{1}{\sqrt{x+9}} \text{ dx.}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series

$$\sum_{1}^{\infty} \frac{1}{\sqrt{n+9}}$$

is convergent or divergent.
Answer: $\boxed{\text{divergent}}$

(f) Evaluate the integral

$$\int_{1}^{\infty} x\ e^{-9x} \text{ dx.}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_{1}^{\infty} ne^{-9n}$ is convergent or divergent.
Answer: $\boxed{10/81e^9}$

(g) Evaluate the integral

$$\int_{1}^{\infty} x\ e^{-9x} \text{ dx}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series

$$\sum_{1}^{\infty} ne^{9n}$$

is convergent or divergent.
Answer: $\boxed{10/81e^9}$

19. (from 7.8 WebAssign, also in Week 7 sample quiz)

(a) Find the values of p for which the integral $\int_{e}^{\infty} \frac{6}{x(\ln x)^p}$ dx converges. Evaluate the integral for these values of p.
(Hint: u-substitution. Check what happens when $p = 1$, when $p < 1$, and when $p > 1$.)
Answer: $\boxed{p > 1 \text{ converges. Otherwise diverges.}}$

(b) Determine whether

$$\int_{2}^{\infty} \left(\frac{1}{e^5}\right)^x \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: $\boxed{1/(5\ 10^e)}$

(c) Determine whether

$$\int_{2}^{\infty} \frac{1}{x^2+8x-9} \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: You can use partial fraction decomposition. $\boxed{\ln(11)/10}$

(d) Determine whether

$$\int_0^1 \frac{4}{x^5}\ \mathrm{dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: $\boxed{\text{divergent}}$

(e) Determine whether

$$\int_0^1 \frac{4}{x^{0.5}}\ \mathrm{dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: $\boxed{8}$

(f) Determine whether

$$\int_2^3 \frac{2}{\sqrt{3-x}}\ \mathrm{dx}$$

is convergent or divergent. If it is convergent, evaluate it.
Answer: $\boxed{4}$