University of Connecticut
Department of Mathematics

Math 1152Q SAMPLE Exam 1 (KEY) Fall 2017

Name: ____________________________________

Read This First !

- This sample is **a lot longer** than the actual test. The actual test will be about 8-9 pages with 1-4 questions on each page.
- From each page, choose a few questions that seem the most difficult.
- You will earn a small amount of bonus 'style points' for a legible, coherent, and non-ambiguous paper. In addition, your reader should not need to reread your solution several times to find a train of thought.
- Please read each question carefully. Show **ALL** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement. No books or other references are permitted.
- All technology (phones, calculators) and books/ notes should be stored inside your bag.

1. For the following questions, circle TRUE or FALSE, and give a justification. True statements should be argued for using facts, theorems or definitions from class. [6]

   (a) If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ converges. **T** **F**

   Justification:
   Answer: False. Counterexample: $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$ and $\sum \frac{1}{\sqrt{n}}$ diverges.

   (b) If $a_n > 0$ and $b_n > 0$ for all n, $\sum b_n$ diverges and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ diverges. **T** **F**

   Justification:
   Answer: False. Counterexample: Let $b_n = \frac{1}{n}$ (and so $\sum b_n$ by p-series/harmonic series test) and $a_n = \frac{1}{n^2}$. We have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2} = 0$. But $\sum a_n$ converges.

   (c) If $a_n > 0$ for all n and $\lim_{n\to\infty} a_{n+1}/a_n = 0$, then $\sum a_n$ is convergent by the ratio test **T** **F**
   Justification:
   Answer: True because $0 < 1$.

   (d) If a_n and b_n are both positive for all n and $\lim_{n\to\infty} a_n/b_n = 0$, then $\sum a_n$ is convergent by the limit comparison test **T** **F**
   Justification:
   Answer: You cannot conclude this in general (for example, when you don't know that $\sum b_n$ converges.

   (e) The harmonic series $\sum 1/n$ is convergent by the p-series test **T** **F**
   Justification:
   Answer: False. The harmonic series is divergent.

   (f) We can use the ratio test alone to show the geometric series $\sum \frac{2^n}{3^n}$ converges **T** **F**
   Justification:
   Answer: True. The ratio of $\frac{2^{n+1}}{3^{n+1}}\frac{3^n}{2^n}$ converges to $\frac{2}{3} < 1$.

   (g) We can use the p-series test alone to show the series $\sum 2^n/3^n$ converges **T** **F**
   Justification:
   Answer: False because $\sum 2^n/3^n$ does not look like a p-series.

(h) We can apply the monotonic sequence theorem to show that the geometric **sequence** $\{2^n/3^n\}_{n=1}^{\infty}$ is convergent **T** **F**
Justification:
Answer: True. The sequence $\{\left(\frac{2}{3}\right)^n\}_{n=1}^{\infty}$ is bounded below (for example, by 0 and -2) and bounded above (for example, by $\frac{2}{3}$ and 5). This sequence is also decreasing. By the monotonic sequence theorem, the sequence is convergent.

(i) We can apply the monotonic sequence theorem to show that the harmonic sequence $\{1/n\}_{n=1}^{\infty}$ is convergent **T** **F**
Justification:
Answer: True. The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is bounded below (for example, by 0 and -2) and bounded above (for example, by 1 and 5). This sequence is also decreasing. By the monotonic sequence theorem, the sequence is convergent.

(j) We can apply the squeeze theorem to show that the harmonic sequence $\left\{\frac{(-1)^n}{n}\right\}$ is convergent **T** **F**
Justification:
Answer: True. Squeeze each term between $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$.

(k) The integration by parts formula can be derived immediately from the product rule. **T** **F**
Justification:
Answer: True.

(l) The integration by parts formula can be derived immediately from the chain rule. **T** **F**
Justification:
Answer: False. See above.

(m) The integration by parts formula can be derived immediately from the u-substitution formula. **T** **F**
Justification:
Answer: False. See above.

(n) It is impossible for a subset of a line to have infinitely many points and have length zero. **T** **F**
Answer: F. The Cantor set, see `http://egunawan.github.io/fall17/hw/problemsAkey.pdf`

2. (a) State the contrapositive of the factual statement: "If the sequence $\{a_n\}$ is unbounded, then it is divergent". [2]
Answer: If the sequence $\{a_n\}$ is convergent, then it is bounded.

(b) Is the contrapositive statement you wrote in part (a) true? **(No justification necessary).** [2]
Answer: True.

3. Answer the following on the line provided. No justification is necessary. [18]

(a) What is the 100th term of the sequence $\{2, 5, 8, 11, \dots\}$?
(The terms 2 and 5 are the first and second term, respectively) ________________

Answer: $-1 + 300 = 299$

(b) Find a formula for the general term a_n of the sequence $\{1, -\frac{2}{5}, \frac{3}{25}, -\frac{4}{125}, \frac{5}{625} \dots\}$.
Make sure to specify your starting value of n.
Answer: Observe that for $n \geq 1$,

$$a_n = \frac{n}{(-5)^{n-1}}$$

(c) Write the geometric series $4 + 2 + 1 + \frac{1}{2} \cdots$ in standard form. ________________
(summation notation)

Answer:

$$\sum_{n=-2}^{\infty} \left(\frac{1}{2}\right)^n$$

(d) Find the 7th term a_7 in the recursive sequence $a_{n+1} = a_n + a_{n-1}$ ________________
when $a_1 = 2$ and $a_2 = 3$.

Answer: $a_7 = 34$

(e) We can use geometric series to compute

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000}.$$

What fraction is this equal to?
Answer: $\frac{1}{9}$.

(f) We can use geometric series to compute $0.9999\dots$. What fraction is this equal to?
Answer: $\frac{1}{1}$.

(g) One of the two decimal expansions for a number is $2.449999\dots$. What's the other?
Answer: 2.45 See # 8 `https://egunawan.github.io/fall17/hw/problemsAkey.pdf`

(h) Use geometric series to compute the fraction for $1.833333\dots$
Answer: $11/6$

(i) Use geometric series to compute the fraction for $1.08333333333333333\dots$
Answer: $13/12$

(j) Does the series $\sum_{n=1}^{\infty} -\ln\left(\frac{n}{2n+7}\right)$ converge or diverge? ________________

Answer: The series diverges by divergence test or limit comparison test.

(k) Find the sum of the series $\sum_{k=0}^{\infty} 5\left(\frac{2}{3}\right)^k$. ________________

Answer: 15

(l) Find the sum of the series $\sum_{n=2}^{\infty} 5\left(\frac{(-6)^{n-1}}{7^n}\right)$. ________________

You don't need to reduce your fraction.

Answer: $-\frac{30}{91}$.

(m) Find the sum of the series ________________

$$-5 + 3 - \frac{9}{5} + \frac{27}{25} - \frac{81}{125} + \dots.$$

Answer: If you want your index to start at $n = 1$, we can have $r = -\frac{3}{5}$ with $a = -5$. Then

$$-5\frac{1}{1-r} = -5\frac{1}{1+\frac{3}{5}} = -\frac{25}{8}.$$

(n) Write an expression for the nth term in the sequence ________________ $\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots\right\}$. (The terms $\frac{1}{2}$ and $\frac{1}{6}$ are the first and second terms in the sequence)

Answer: $a_n = \frac{1}{(n+1)!}$

(o) Write an equivalent series with index summation beginning at $n = 0$. ________________ $\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!}$

Answer: $\sum_{n=0}^{\infty} \frac{2^{n+2}}{(n)!}$

(p) For what values of k does the series $\sum \frac{5}{n^k}$ converge? ________________

Answer: $k > 1$

4. The following questions ask you to determine the converge/divergence of a series. [5]

   (a) Use the **limit comparison test** to determine whether the series $\sum_{n=3}^{\infty} \dfrac{6}{n\sqrt{n^2-8}}$ converges or diverges.
   Answer: Let $a_n = \dfrac{6}{n\sqrt{n^2-8}}$ Let $b_n = \dfrac{1}{n^2}$. Then $\dfrac{a_n}{b_n} \to 6$ (which is a positive number) as $n \to \infty$. Since $\sum b_n$ converges by the p-series test, we conclude that $\sum a_n$ converges by limit comparison test.

   (b) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{\sqrt{n^3+1}}{3n^3-4n+2}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   Answer: Let $b_n = \dfrac{1}{n^{3/2}}$. Note that $\sum b_n$ converges since it's a p-series with $p = 3/2 > 1$. Moreover,
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{\sqrt{3}} > 0.$$
   Therefore, since $\sum b_n$ converges, $\sum a_n$ also converges by the Limit Comparison Test.

   (c) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{n!}{7^n(n+8)!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   Answer: The series converges. Both the ratio test and limit comparison test work.

   (d) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(n+8)!}{7^n n!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   Answer: The series converges. Both the ratio test and limit comparison test work.

   (e) Determine whether the series $\sum_{n=1}^{\infty} \ln\left(\dfrac{3n}{n+1}\right)$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   Answer: The series diverges by the Divergence Test. Another test that would work is the limit comparison test.

   (f) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{5n+3}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
   Answer: Let $b_n = \dfrac{1}{5n+3}$. Then
   $$b_{n+1} \le b_n \text{ for all } n \ge 1$$
   and $\lim_{n\to\infty} b_n = 0$. Therefore the series converges by the Alternating Series Test.

5. Find the values of A so that the geometric series $\sum_{n=1}^{\infty} \dfrac{(A-3)^{n-1}}{3^{n-1}}$ is convergent. [8]

   Answer: $0 < A < 6$ by the geometric series test.

6. (a) Compute $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{3n}$. [5]

Answer: e^3

(b) Compute $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{2n}\right)^{n}$.

Answer: $\sqrt{e}$

(c) Compute $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^{5n}$.

Answer: e^{10}

(d) Compute $\lim_{n\to\infty} n^2e^{-n}$.

Answer: Use L'Hopital rule twice to get 0.

(e) Compute $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

Answer: Use L'Hopital's rule once to get 0.

(f) Compute $\lim_{n\to} \dfrac{n\sin n}{n^2+1}$.

Answer: Observe that

$$-\frac{n}{n^2+1} \le \frac{n\sin n}{n^2+1} \le \frac{n}{n^2+1}.$$

Since $\lim_{n\to\infty} -\dfrac{n}{n^2+1} = 0 = \lim_{n\to\infty} \dfrac{n}{n^2+1}$, by the squeeze theorem we can conclude that $\lim_{n\to\infty} \dfrac{n\sin n}{n^2+1} = 0$.

7. Consider the series $\sum a_n = \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n-1}$.

   (a) What are the first three terms in the series? ________________ [3]

   Answer: $\frac{\cos(2\pi)}{2-1} = \frac{1}{1}$, $\frac{\cos(3\pi)}{3-1} = -\frac{1}{2}$, $\frac{\cos(4\pi)}{4-1} = \frac{1}{3}$

   (b) Is the series convergent? You must justify. ________________ [5]

   Answer: The series converges by the the alternating series test.

   (c) Is the series $\sum a_n = \sum_{n=2}^{\infty} \frac{|\cos(n\pi)|}{n-1}$ convergent? You must justify. ________________ [5]

   Answer: The series diverges by limit comparison test with $b_n = \frac{1}{\sqrt{n}}$ or by direct comparison test with $b_n = \frac{1}{n-1}$.

8. (a) Evaluate $\lim_{n\to\infty} e^{-n}\sqrt{n}$. [5]
   Answer: 0.

   (b) Determine whether $\sum_{n=0}^{\infty} e^{-n}\sqrt{n}$ converges or diverges.
   Answer: The series converges.
   You can use limit comparison test (compare the terms with any geometric sequence with ratio between $\frac{1}{e}$ and 1 like $b_n = \frac{1}{2^n}$ OR any p-sequence $b_n = \frac{1}{n^p}$ where $p > 1$).
   You can also use the ratio test.

   (c) Evaluate $\lim_{n\to\infty} \frac{(\ln(n))^2}{n^2}$.
   Answer: 0.

   (d) Determine whether $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^2}$ converges or diverges.
   Answer: The series converges .
   You can use limit comparison test (compare the terms with any p-sequence $b_n = \frac{1}{n^p}$ where $1 < p < 2$).
   Non-comparison tests do not work.

   (e) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with the property that

   $$a_1 + a_2 + \cdots + a_n = 2 - 3(0.8)^n.$$

   State whether $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, find its sum.
   Answer: The expression above is the nth partial sum

   $$S_n = 2 - 3(0.8)^n.$$

   By definition, the series converges to

   $$\lim_{n\to\infty} S_n = 2.$$

9. Compute the following definite and indefinite integrals.

(a) $\int_1^2 t^3 \ln(t) dt$ [5]

Answer: $\ln(16) - \frac{15}{16}$. The indefinite integral is $\frac{1}{4}(t^4 \ln(t) - \frac{t^4}{4})$.

(b) $\int \frac{2x-3}{8+x^2}$ dx [5]

Answer: Draw a triangle and use inverse trig substitution. Then use u-substitution where $u = \cos(\theta)$ and $du = -\sin(\theta)$. Then you get $\int \frac{2x-3}{8+x^2}$ dx $= ln(x^2+8) - \frac{3}{2\sqrt{2}} \arctan\left(\frac{x}{2\sqrt{2}}\right)$.

(c) $\int \frac{\sin(\ln(x))}{x}$ dx [5]

Answer: Use u-substitution with $u = \ln(x)$ and $du = \frac{1}{x}$ dx. Then $\int \frac{\sin(\ln(x))}{x}$ dx $= -\cos(\ln(x)) +$ Constant.

(d) $\int_0^1 x\ e^{-x^2}$ [5]

Answer: Use u-substitution with $u = -x^2$ and $du = -2x$ dx. Then $\int_0^1 x\ e^{-x^2} = -\frac{1}{2} e^u \Big|_0^{-1} = \frac{1}{2}(1 - e^{-1})$. Sanity check: your answer should be positive because xe^{-x^2} is positive on the interval $(0, 1]$.

(e) $\int (x+2) \sin(3x)$ dx [5]

Answer: Integration by parts with $u = x + 2$ and $dv = \sin(3x)$ dx. Then $\int (x + 2) \sin(3x)$ dx $= -\frac{1}{3}(x+2)\cos(3x) + \frac{1}{9}\sin(3x) +$ Constant.

(f) $\int e^{\sqrt{x}}$ dx [5]

Answer: Do substitution with $w = \sqrt{x}$ and $dw = \frac{1}{2} x^{-\frac{1}{2}}$ dx. Then do integration by parts with $u = w$ and $dv = e^w$ dx. Then $\int e^{\sqrt{x}}$ dx $= 2we^w - 2\int e^w dw = 2we^w - 2e^w = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} +$ Constant.

(g) $\int \frac{x}{\sqrt{4-x^2}}$ [5]

Answer: u-substitution is the easiest. $\int \frac{x}{\sqrt{4-x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}} du = -u^{1/2} + C = -\sqrt{4-x^2} + C$.

10. (a) Sketch the graph of each function and shade the region whose area is represented by the integral below. **Label all pertinent information.** [5]

$$\int_{-3}^{4} (2x + 15) - x^2 \text{ dx}$$

Do not evaluate.
Answer: Checking your graph using Desmos `https://www.desmos.com/calculator/vem9dirors`.

(b) Consider the region bounded by [5]

$$y = x^2, \ y = 2 - x^2.$$

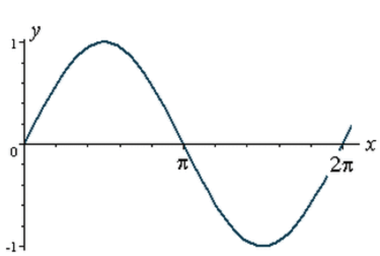
1. Find the intersection points of the two curves.
   Answer: -1 and 1.

2. Sketch the two curves and shade the region bounded by the two curves.
   Answer: Use Desmos to verify your graph.

3. Set up, but **do not evaluate** an integral for the area of the shaded region.
   Answer:

$$\int_{-1}^{1} 2 - 2x^2 \text{ dx}$$

11. Evaluate the area of the region bounded by the curves $y = \sin(x)$, $y = 0$, $x = 0$ and $x = \dfrac{\pi}{2}$. [5]
For your convenience, the graph of $y = \sin(x)$ is shown below.



Answer: 1

12. Consider the following integrals and decide whether the best method of integration is integration by parts, u-substitution, or trig substitution. Explain the first key step/s of evaluating the integrals (There often are more than one right answer). **Do not evaluate the integrals.**

(a) $\int_0^4 \frac{\ln(x)}{\sqrt{x}}$ dx ______________________ [2]

Answer: Integration by parts with $u = \ln(x)$ and $dv = \frac{1}{\sqrt{x}}$.

(b) $\int \frac{1}{x\ln(x)}$ dx ______________________ [2]

Answer: u-substitution for $\ln(x)$.
At the end you should get $\ln(\ln(x))+$ Constant.

(c) $\int_1^2 \ln(x)$ dx ______________________ [2]

Answer: Integration by parts with (the only option) $u = \ln(x)$ and $dv =$ dx

(d) $\int xe^{0.2x}$ dx ______________________ [2]

Answer: Integration by parts with $u = x$ and $dv = e^{0.2x}$ dx

(e) $\int_0^1 e^x \sin(x)$ dx ______________________ [2]

Answer: Integration by parts with either $u = e^x$ and $dv = \sin(x)$ dx OR $dv = e^x$ dx and $u = \sin(x)$

(f) $\int \frac{1}{x^2+2x+4}$ dx ______________________ [2]

Answer: Complete the square, then use inverse trig substitution with $x+\sqrt{3} = tan(\theta)$ or $x+\sqrt{3} = \cot(\theta)$

13. Do the step-by-step work to show that

$$\int \frac{1}{x^2+25}\,\text{dx} = \frac{1}{5}\arctan\left(\frac{x}{5}\right) + c.$$

Answer: Do inverse trig substitution $x = 5\tan(\theta)$ so that you get

$$\int \frac{1}{x^2+25}\,\text{dx} = \int \frac{1}{5}\theta\, d\theta$$