**Last updated on 2016/12/06 at 14:05:40. Please inform me of mistakes.**

# 1 Materials covered, and resources

This test covers lecture days Nov 14 – Dec 5, and that means textbook sections 4.4 (composition and derivatives), 4.5 (Implicit differentiation), 4.6 (Related rates), and 5.1 (Extreme Value Theorem, critical numbers, extrema), 5.3 (Increase/decrease/max/min), and 5.4 (Concavity and second derivatives). Note that we skipped Sec 5.2 (Mean Value Theorem) because we did it earlier this semester.

You should find that the course website is all up to date with solutions to everything from those days, including in-class exercises, quizzes, problems, and so on. If you notice anything missing (or incorrect), let me know.

You should be familiar with the following list of CONCEPTS:

- Composite Functions - Chain rule
- Theorem 5.1 (Extreme Value Theorem)
- Theorem 5.5 (Increasing/Decreasing Test)
- Theorem 5.6 (First Derivative Test for relative Maxes and Mins)
- Concave up, concave down, and inflection points
- Theorem 5.7 (Test for concavity)

And practice specific SKILLS:

- Compute derivatives with chain rule (and other derivative rules)
- Apply implicit differentiation to plane curves
- Apply implicit differentiation to related rates problems
- Find extreme values (maxes and mins) on a closed interval
- Find critical numbers, identify intervals of increase and decrease
- Classify critical numbers (relative maxes? relative mins?)
- Find inflection points and intervals of concavity

You pretty much know what to expect by now - it won't be radically different in style from any of the other regular midterm tests so far. Practice on suggested textbook exercises, and textbook reading.

## 2 Topic: Sec 4.4 Composition of functions, Chain Rule (plus all the previous derivative rules from Chapter 4)

1. Write $h(x) = \frac{1}{x+1}$ as a composition of two simpler functions. Then compute $h'(x)$ using Chain Rule.
   Ans: See Sec 4.4 Example 2a p. 286

2. Write $h(x) = \sqrt{3x^2 - x + 1}$ as a composition of two simpler functions. Then compute $h'(x)$ using Chain Rule
   Ans: See Sec 4.4 Example 2b p. 286

3. Use derivative rules to find $f'(x)$. On the test, do not spend a lot of time simplifying once the differentiation steps are completed.
   (a) $f(x) = (x^2 + 1)^3$. Check your answer with WolframAlpha.
   Ans: See Sec 4.4 Example 3 p. 287
   (b) $f(x) = \frac{-7}{(2x-3)^2}$. Check your answer with WolframAlpha.
   Ans: See Sec 4.4 Example 6 p. 287
   (c) $f(x) = \sqrt[3]{(x^2 - 1)^2}$. Check your answer with WolframAlpha.
   Ans: See Sec 4.4 Example 5 p. 287

4. Use your answer from above to find the critical numbers of $f(x) = \sqrt[3]{(x^2 - 1)^2}$. For definition of critical numbers, see Sec 5.1.
   Ans: The critical numbers of $f$ are $x = -1$, $x = 0$, and $x = 1$. See Sec 4.4 Example 5 p. 287.

5. Read all of Sec 4.4, p284-289. Work through Examples 2-6 (chain rule). Work through Examples 6-9 (how to simplify after differentiation in a clear, readable way).

6. Book Exercises: Sec 4.4 #1-12 (odd numbers first); 47, 59, 60, 61.

## 3 Topic: Sec 4.5 Implicit Differentiation

1. Find $dy/dx$ given that $y^3 + y^2 - 5y - x^2 = -4$. (Hint: Differentiate both sides with respect to $x$, then solve for $dy/dx$).
   Ans: See Sec. 4.5 Example 2 p. 293.

2. Find the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $(\sqrt{2}, \frac{-1}{\sqrt{2}})$. (Hint: Differentiate both sides with respect to $x$, then solve for $dy/dx$, then get the slope you need).
   Ans: $\frac{1}{2}$. See Sec 4.5 Example 4 p. 294.

3. Find the equation of the tangent line to the graph of $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$. (Hint: Differentiate both sides with respect to $x$, then solve for $dy/dx$, then get the slope you need, then write the equation of the line).
   Ans: $y = 3x - \sqrt{2}$. See Sec 4.5 Example 8 p. 296.

4. Read Sec 4.5 p292-296 for many patiently-worked out implicit diff examples; you'll need to work through them yourself to get fluent with the process.

5. Book Exercises: After you do the reading and work the examples in the section, try Sec 4.5 p297 #1,4, 5, 15, 21.

# 4 Topic: Sec 4.6 Related Rates

1. Suppose that $x$ and $y$ are both differentiable functions of $t$, and they are related by the equation $y - x^2 = 3$. Find $dy/dt$ when $x = 1$, given that $dx/dt = 2$ when $x = 1$.
   Ans: 4. See Sec. 4.6 Example 1 on p. 300.

2. I have a 10-ft ladder that is leaning against a wall. At this particular moment, the distance between the bottom of the ladder and the wall is 8 feet. The bottom of the ladder is sliding away from the wall at the rate of 4 ft/second. Hence the top of the ladder is sliding down along the wall. Compute the rate of which the top of the ladder is sliding down at a particular time $t$. (Hint: See the Falling ladder related rates from Khan Academy).
   Ans: The top of the ladder is sliding down at 16/3 feet per second.

3. Read Sec 4.6 p300-302. Read the exposition and work through Examples #1-3.

4. Book Exercises: Sec 4.5 #25-31 (odd), 33a, 47; Sec 4.6 #15 (Hint: see Example 2 in the section).

5. Read book Sec 4.6 Ex. 4 (p301).

6. Rework Examples 1-5 (p300-303) on your own (solve it without looking at the book, and then compare to the textbook solution).

# 5 Topic: Sec 5.1 Extreme Value Theorem, Extrema

1. Give a complete and accurate statement of the Extreme Value Theorem.
   Ans: See Thm 5.1 p. 314.

2. Complete the definition. Let $f$ be a function which is defined on $c$. Then we say that $c$ is a *critical number* of $f$ if ....
   Ans: See Sec 5.1 p. 316 (def of critical number).

3. True or False: If 5 is a critical number of $f$, then $f$ has a relative minimum or relative maximum at $x = 5$.
   Answer: False. Counterexample: $f(x) = (x - 5)^3$.

4. True or False: If $f$ has a relative maximum at $x = 2$, then 2 is a critical number of $f$.
   Answer: True, by Thm 5.2 p. 316 from Sec 5.1.

5. Find the absolute minimum and absolute maximum of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.
   Ans: $-1$ and 16. See Sec 5.1 Example 2 on p. 317.

6. Answer True or False.

   (a) The maximum of a function that is continuous on a closed interval can occur at only one value in the interval.
   Ans: False. Counterexample, the maximum of $x^2$ occurs twice (at $x = -3$ and $x = 3$) on the closed interval $[-3, 3]$.

   (b) If a function is continuous on $[-3, 3]$, then it must have a minimum on the interval.
   Ans: True, by Extreme Value Thm.

(c) If $x = 5$ is a critical number of the polynomial $p(x)$, then $x = 5$ must also be a critical number of the polynomial $f(x) = p(x) + 7$.
Ans: True, since $f'(5) = p'(5)$ if it is defined. If $p'(5)$ is not defined, then $f'(5)$ is also not defined.

(d) Read Sec 5.1 p314-318. Work through Ex. 2 and 3 on p317-318 (meaning, after you have read through them, copy the problem out and rework it on your own to see if you've really understood and remembered the process).

(e) Book Exercises: Sec 5.1 (p319) #1-13 - most of these are answered just by looking at a graph, without any calculating. But you'll need to understand the terms and definitions from the reading!

(f) More Book Exercises: Sec 5.1 p319 #14,16, 17, 19, 21, 45, 46, 47-50 (many of these are graphical exercises that don't require a lot of calculation)

# 6 Topic: Sec 5.3 Increase/Decrease/Min/Max

1. Let $f(x) = x^3 - \frac{3}{2}x^2$.

   (a) What are the critical numbers of $f$?
   Ans: 0 and 1

   (b) Determine the intervals of increase and decrease for $f$.
   Ans: See Sec 5.3 Example 1 on p. 330.

   (c) Classify the critical numbers (relative max/ relative min/ neither).
   Ans: $f$ has a local max at the point where $x = 0$, and $f$ has a local min at the point where $x = 1$. See the explanation at the top of p. 331.

2. Let $f(x) = 2x^3 - 3x^2 - 36x + 14$.

   (a) What are the critical numbers of $f$?
   Ans: $-2$ and 3

   (b) Determine the intervals of increase and decrease for $f$.
   Ans: See Sec 5.3 Example 2 on p. 332.

   (c) Classify the critical numbers (relative max/ relative min/ neither).
   Ans: $f$ has a local max at the point where $x = -2$, and $f$ has a local min at the point where $x = 3$. See Sec 5.3 Example 2 on p. 332.

3. Find the relative maximum and minimum of

$$f(x) = \frac{x^4 + 1}{x^2}.$$

   Ans: See Sec 5.3 Example 4 on p. 334.

4. Answer True of False

   (a) The sum of two increasing functions is increasing.
   Ans: True

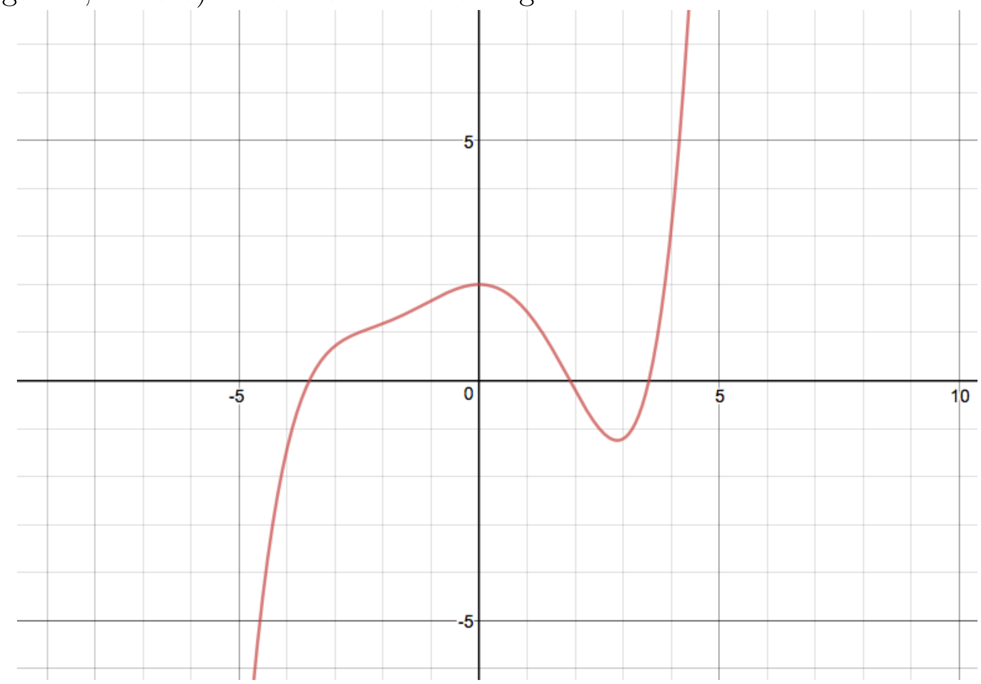
   (b) Every $n$th degree polynomial has $(n - 1)$ critical numbers.
   Ans: False. For example, how many critical numbers does $x^5$ has?

(c) There is a relative maximum or minimum at each critical number.
Ans: False. For example, $x^5$ has $x = 0$ as a critical number, but $x^5$ does not have a relative maximum or minimum at $x = 0$.

5. Read: Sec 5.3 p329-334.
6. Memorize statements of Theorems 5.5 and 5.6.
7. Work through Examples 1-4 (read, understand, copy the problem, and rework on your own)
8. Book Exercises: Sec 5.3 p335 #1,3,8,11,15,17,19

# 7 Figuring out signs from graphs (Secs. 5.1,5.2,5.4)

The figure shows the graph of a function $f(x)$. Use the graph to determine the sign (positive, negative, or zero) of each of the following.




1. f(-4)
Answer: negative
2. f(-3)
Answer: positive
3. f(0)
Answer: positive
4. f(1)
Answer: positive
5. f(3)
Answer: negative
6. $f'(-4)$
Answer: pos
7. $f'(-3)$
Answer: pos
8. $f'(0)$
Answer: 0
9. $f'(1)$
Answer: neg
10. $f'(4)$
Answer: pos
11. $f''(-4)$
Ans: neg
12. $f''(-3.5)$
Ans: neg
13. $f''(0)$
Ans: neg
14. $f''(1)$
Ans: neg
15. $f''(3)$
Ans: pos

# 8 Topic: Sec 5.4 Concavity

1. Complete the definition. Let $f$ be differentiable on an open interval $I$.
   a. The graph of $f$ is *concave up on* $I$ means that ....
   b. The graph of $f$ is *concave down on* $I$ means that ....
   Ans: See defn in Sec 5.4 top of p. 338

2. Give a complete and accurate statement of the test for concavity (Thm 5.7 on p. 339).
   Ans: See Sec 5.4 Thm 5.7 Test for Concavity.

3. Determine the open intervals on which the graph of $\frac{6}{x^2+3}$ is concave up or down.
   Ans: Concave up on $(-\infty, -1)$ and $(1, \infty)$. Concave down on $(-1, 1)$. See Sec 5.4 Example 1 on p. 339.

4. Suppose $f$ is a differentiable function on $(-\infty, \infty)$. Answer True or False.

   (a) If $f''(x)$ is positive on an interval $(a, b)$, then $f(x)$ is concave up on $(a, b)$.
   Ans: True, by Thm 5.7 Test for Concavity

   (b) If $f''(x)$ is negative on an interval $(a, b)$, then $f(x)$ is concave down on $(a, b)$.
   Ans: True, by Thm 5.7 Test for Concavity.

   (c) If $f''(x)$ is positive on an interval $(a, b)$, then $f(x)$ is increasing on $(a, b)$.
   Ans: False. Counterexample: Consider $f(x) = x^2$ on the interval $(-5, 5)$. Then $f''(x) = 2$, so $f''(x)$ is positive for all real $x$ in $(-5, 5)$, but $f$ is not increasing on $(-5, 5)$.

   (d) If $f''(x)$ is negative on an interval $(a, b)$, then $f(x)$ is decreasing on $(a, b)$.
   Ans: False. Counterexample: Consider $f(x) = -x^2$ on the interval $(-5, 5)$. Then $f''(x) = -2$, so $f''(x)$ is negative for all real $x$ in $(-5, 5)$, but $f$ is not decreasing on $(-5, 5)$.

5. If $f$ is continuous everywhere, and $f$ changes from being concave up to concave down at a point where $x = 5$, then this point (where $x = 5$) is a point of inflection of the graph $f$.
   Ans: True, by definition of inflection point. See Sec 5.4 on p. 340.

6. Suppose $f(x) = x^N$ where $N$ is an odd positive number larger than 1.

   (a) Does $f$ have any inflection point/s? How many?
   Ans: Just one, at $x = 0$.

   (b) Determine the intervals on which the graph of $f$ is concave up or down.
   Ans: concave down on the interval $(-\infty, 0)$, and concave up on the interval $(0, \infty)$.

7. Suppose $g(x) = x^N$ where $N$ is an even positive number.

   (a) Does $g$ have any inflection point/s? How many?
   Ans: None

   (b) Determine the intervals on which the graph of $g$ is concave up or down.
   Ans: concave up on $(-\infty, \infty)$.

8. Answer True or False. These are exercises # 65-68 from p. 345.

(a) Consider a cubic polynomial $h(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. This cubic polynomial has precisely one point of inflection.
Ans: True. Explanation: the first derivative of $h$ is $3ax^2 + 2bx + c$, so the second derivative of $h$ is $6ax+2b$. But $6ax+2b = 0$ if $x = -\frac{b}{3a}$; $6ax+2b$ is positive whenever $x > -\frac{b}{3a}$; and $6ax + 2b$ is negative whenever $x < -\frac{b}{3a}$.

(b) If $f'(c) > 0$, then $f$ is concave upward at $x = c$.
Ans: False.

(c) If $f''(2) = 0$, then the graph of $f$ must have a point of inflection at $x = 2$.
Ans: False. For example, if $f(x) = x$, the graph have no inflection point even though $f''(2) = 0$.

9. Consider a cubic polynomial $h(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. Determine the intervals on which the graph of $h$ is concave up or down.
Ans: Concave down on $(-\infty, -\frac{b}{3a})$. Concave up on $(-\frac{b}{3a}, \infty)$. See the explanation above for the T/F question.

10. Read Sec 5.4 p338-341. Work the examples and study the figures that go along with them!

11. Book Exercises: Sec 5.4 #1,2,3 (solve these analytically using the second derivative, but look at the graph that's provided to see if your answer matches), #9, 11, 13, 15, 37 and 38, 65.