Last updated on 2016/11/10 at 14:22:53. Please inform me of mistakes.

1 Materials covered, and resources

This test covers lecture days Oct 20 – Nov 8, and that means textbook sections 4.1, 4.2, 4.3 on derivatives, and (skipping a bit) section 5.2 on Rolle's Theorem and the Mean Value Theorem.

You should find that the course website is all up to date with solutions to everything from those days, including in-class exercises, quizzes, problems, and so on. If you notice anything missing (or incorrect), let me know.

Here are a couple of lists that I jotted down as I was preparing the test:

Concepts:

- Derivative as slope,
- Derivative as velocity (2nd Derivative as acceleration),
- Derivative Rules,
- Differentiability vs continuity,
- expanding powers of binomials / proof of the power rule,
- derivatives of polynomials, derivatives of rational functions, constant multiple, sum/difference, product, and quotient rules.

Skills:

- Compute a derivative using the limit definition,
- compute derivatives of polynomials, rational functions, roots and other power functions using derivative rules,
- construct tangent lines,
- find turning points of a polynomial function,
- find second/third/etc. derivatives.

Practice on suggested textbook exercises, and textbook reading.

2 Topic: the derivative function as a limit

Given the following functions for $f(x)$, use the *definition* (limit in sec 4.1) of derivative to evaluate $f'(x)$.

1. $f(x) = \sqrt{x+5}$
   Answer: $\frac{1}{2\sqrt{x+5}}$
2. $f(x) = \sqrt{x+3}$
   Answer: $\frac{1}{2\sqrt{x+3}}$
3. $f(x) = \sqrt{x+1}$
   Answer: $\frac{1}{2\sqrt{x+1}}$
4. $f(x) = 2x^3 + 1$

3 Tangent lines

Suppose that $f(x) = \frac{-7}{(2x-3)^2}$ and I computed for you $f'(x) = \frac{28}{(2x-3)^3}$.

1. Use the given information to find the slope of the tangent line to the graph of f at $x = 5$
   Answer: $\frac{28}{343}$ or $\frac{4}{49}$

2. Write an equation for the tangent line at $x = 5$.
   Answer: $(y - f(5)) = \frac{4}{49}(x - 5)$ which is also $(y + \frac{1}{7}) = \frac{4}{49}(x - 5)$

4 Applying the power rule, constant multiple,product rule, and quotient rule. Plus graphical applications of derivatives.

0a. How do you explain why the power rule holds? Look back to the Handout:*Expanding Powers of Binomials [PDF]* from October 31 (Monday) lecture.
0b. How do you explain why the product rule holds? Look back to lecture Nov 4 (Friday).

Compute the derivative of each of the following functions. Use the derivative rules. Do not set up a limit for these.

1. $f(x) = 3x^{10}$
   Answer:$30x^9$ by power rule and constant multiple

2. $f(x) = 3x^{-10}$
   Answer:$-30x^{-11}$ by power rule and constant multiple

3. $f(x) = -\frac{1}{3}x^{-10}$
   Answer:$\frac{10}{3}x^{-11}$ by power rule and constant multiple

4. $f(x) = -\frac{1}{x^3}$
   Answer: $\frac{3}{x^4}$ by power rule and constant multiple

5. $f(x) = \frac{42}{x^{10}}$
   Answer: $\frac{-420}{x^{11}}$ by power rule and constant multiple

6. $f(x) = \sqrt[7]{x}$
   Answer: $\frac{1}{7}x^{-6/7}$ by power rule

7. $f(x) = \sqrt[7]{2}$
   Answer: 0 because $\sqrt[7]{2}$ is a constant.

8. $f(x) = \sqrt[7]{2}\,x$
   Answer: $\sqrt[7]{2}$ by power rule and constant multiple

9. $f(x) = \frac{5x-2}{x^2+1}$
   Answer: Use the quotient rule to get $\frac{-5x^2+4x+5}{(x^2+1)^2}$. Sec 4.3 Example 3.

10. a.) $f(x) = \frac{3-(1/x)}{x+5}$
Answer: Use the quotient rule to get $f'(x) = \frac{-3x^2+2x+5}{(x^2+5x)^2}$. See Sec 4.3 Example 4.

b.) After that, find the points (x-values) where the graph of the function has a horizontal tangent line.
Answer: If there is a horizontal tangent line at x then $-3x^2 + 2x + 5 = 0$.
But$-3x^2 + 2x + 5 = -(3x-5)(x+1)$, so either $x = -1$ or $x = \frac{5}{3}$.

c.) Use your answer from above to find an equation of the tangent line to the graph $f(x) = \frac{3-(1/x)}{x+5}$ at the point $(-1, 1)$.
Answer: $y = 1$.

5 Higher derivatives

1. Compute the second derivative $\frac{d^2}{dx^2}(3x^{10} + x^2)$
Answer: $\frac{d}{dx}(30x^9) = 270x^8 + 2$

2. Compute the third derivative $\frac{d^3}{dx^3}(3x^{10} + 5x^2)$
Answer: $\frac{d^2}{dx^2}(30x^9) = \frac{d}{dx}(270x^8) = 2160x^7$

6 Knowing derivative rules

1. Suppose f is a differentiable function with respect to x. Apply one of the derivative rules rules and write down the derivative of $5\,f(x)$.
Answer: $5\,f'(x)$

2. Suppose f is a differentiable function with respect to x. Apply the product rule, and write down the derivative of $x^2\,f(x)$.
Answer: $x^2\,f'(x) + f(x)\,2x$

3. Suppose f is a differentiable function with respect to x. Apply the quotient rule, and write down the derivative of $\frac{f(x)}{2x^5}$.
Answer: $\frac{(2x^5)f'(x) - f(x)(10x^4)}{4x^{10}}$

7 Figuring out from a graph whether f is continuous, not continuous, and differentiable

1. Sketch an example of a graph that is differentiable everywhere except at $x = 1$ and $x = 8$.
Answer: see p258 (sec 4.1) in textbook Figures 4.11-4.13

2. Answer True or False. If a graph has a vertical tangent line at $x = 2$, then this graph is differentiable at $x = 2$.
Answer: False. See Example 7 p258 (sec 4.1).

3. Answer True or False. If a graph has a horizontal tangent line at $x = 2$, then this graph is differentiable at $x = 2$.
Answer: True. The derivative exists at $x = 2$ (and is equal to 0).

4. Answer True of False. If f is differentiable at $x = 5$, then f is continuous at $x = 5$.
   Answer: True by Theorem 4.1

5. Answer True of False. If f is not continuous at $x = 3$, then f is not differentiable at $x = 3$.
   Answer: True by (the contrapositive of) Theorem 4.1

6. Answer True of False. If f is continuous at $x = 2$, then f is differentiable at $x = 2$.
   Answer: False. For example the function $f(x) = |x - 2|$ is continuous at $x = 2$, but the it is not differentiable at $x = 2$ (see Example 4 sec 4.1, p258).

7. Suppose f is a function. What does it mean for f to be differentiable at $x = c$?
   Answer: f is differentiable at $x = c$ if

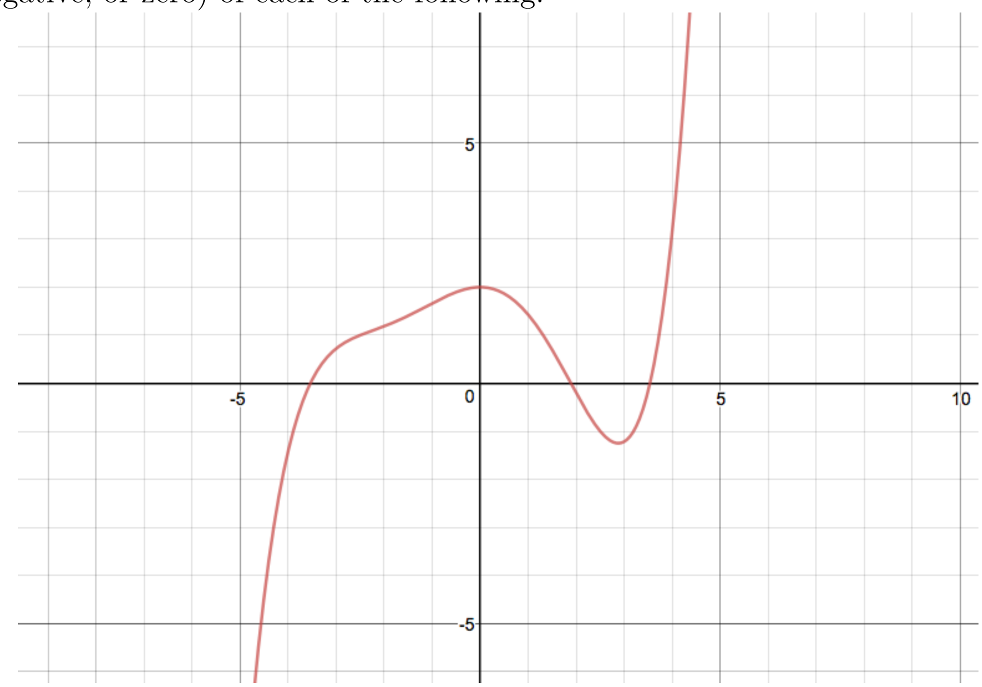
$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(c)}{\Delta x}$$

   exists.
   See Sec 4.1 p255 Definition of the derivative of a function.

8. What does it mean for f to be continuous at $x = c$? Write a precise definition.
   Answer: Three conditions need to be met. See definition of continuity on p227 in Sec 3.4 textbook.

8 Topic: Figuring out signs from graphs

The figure shows the graph of a function $f(x)$. Use the graph to determine the sign (positive, negative, or zero) of each of the following.



1. f(-4). Answer: negative
2. f(-3) Answer: positive
3. f(0) Answer: positive
4. f(1) Answer: positive

5. f(3) Answer: negative
6. f'(-4). Answer: pos
7. f'(-3) Answer: pos
8. f'(0) Answer: 0
9. f'(1) Answer: neg
10. f'(4) Answer: pos

9 Rolle's Theorem and Mean Value Theorem

1. Fill in the blanks so that the following becomes a statement of Rolle's Theorem:
Suppose f is **differentiable** on the interval $[a, b]$, and

$$f(a) = \mathbf{f(b)}.$$

Then there is at least one $\mathbf{c}$ in $(\mathbf{a}, \mathbf{b})$ where $\mathbf{f'(c) = 0}$.

2. Fill in the blanks so that the following becomes a statement of the Mean Value Theorem (MVT):
Suppose f is **differentiable** on the interval $[\mathbf{a}, \mathbf{b}]$. Then there is at least one $\mathbf{c}$ in $(\mathbf{a}, \mathbf{b})$ such that

$$\mathbf{f'(c)} = \frac{\mathbf{f(b) - f(a)}}{\mathbf{b - a}}.$$

10 Mean Value Theorem Practice

1. Let $f(x)$ be a function that is differentiable on an interval $[2, 5]$, and $f(2) = f(5) = -1$.

a. True or false. There is some c between 2 and 5 such that the tangent line to the graph f at $x = c$ is horizontal.
Answer: True, by Rolle's Theorem/Mean Value Theorem (sec 5.2)

b. True or false. There is some c between 2 and 5 such that the tangent line to the graph f at $x = c$ is vertical.
Answer: False. Counterexample, consider the constant function $f(x) = -1$.

c. True or false. There is some c between 2 and 5 such that the tangent line to the graph f at $x = c$ has slope -1.
Answer: False. Counterexample, consider the constant function $f(x) = -1$.

2. Suppose you left Saint Peter at 9am and drove to your friend's house 100 miles away. You arrived at 11am, so you completed the trip in two hours.

a. What is your *average speed* for the entire trip?
Answer: $50mph$ (because you drove 100 miles in 2 hours).

b. Sometime between 9 and 11am, did your speedometer ever read exactly 50 mph? In other words, was your *instantaneous speed* ever exactly 50 mph? Yes or no, and why?
Answer: yes, due to Mean Value Theorem (sec 5.2)

3. Suppose f is differentiable on the interval $[1, 7]$. You also know that $f(1) = -3$, $f(6) = 5$, and $f(7) = 15$. Answer True or False. (If false, give a counterexample. If true, explain).

a. _____ f must also be continuous on the interval $[1,7]$.
Answer: True, by Theorem 4.1 (textbook)

b. _____ There must be at least one number c in $(1,7)$ where $f'(c) = 3$.
Answer: True, due to MVT, since $\frac{f(7)-f(1)}{7-1} = 3$

c. _____ There must be at least one number c in $(6,7)$ where $f'(c) = 0$.
Answer: False. You can sketch a differentiable function that is increasing on $(6,7)$. There is no point between 6 and 7 where the graph of this function has a horizontal tangent line.

11 Functions and their derivatives

Answer True or False (If false, give a counterexample. If true, explain):

1. T/F: The derivative of a polynomial must be a polynomial.
Answer: True, due to the power rule, sum rule, and constant rule for derivatives.

2. T/F: The derivative of a rational function must be a rational function.
Answer: True. Explanation: Suppose $r(x)$ is a rational function. That is, by definition of rational functions, $r(x) = f(x)/g(x)$ for some polynomials $f(x)$, $g(x)$ where $g(x) \neq 0$. By the quotient rule, we know that $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$. We know that $f'(x)$ and $g'(x)$ are both polynomials because derivatives of polynomials are polynomials (see the previous question), so $g(x)f'(x) - f(x)g'(x)$ is a polynomial. Therefore, $\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$ is a polynomial.

3. T/F: The derivative of an even-degree polynomial must also be an even-degree polynomial.
Answer: False. Counterexample: The derivative of x^2 with respect to x is $2x$, which is an odd-degree polynomial.

4. T/F: The derivative of an odd-degree polynomial must also be an odd-degree polynomial.
Answer: False. Counterexample: The derivative of x^3 with respect to x is $3x^2$, which is an even-degree polynomial.

5. T/F: The derivative of an odd-degree polynomial must be an even-degree polynomial.
Answer: True, because $\frac{d}{dx}(x^{2k+1}) = (2k+1)x^{2k}$, and $2k$ is even.

6. T/F: The derivative of an even-degree polynomial $f(x)$ (where the degree is 2 or higher) must have at least one zero).
Answer: True. Explanation: The derivative of $f(x)$ must be a polynomial of odd degree. We learned about the end behaviors of an odd-degree polynomial (during the first month of the semester). It either falls to the left and rises to the right, or it rises to the left and falls to the right.

7. T/F: The derivative of an odd-degree polynomial $f(x)$ must have at least one zero.
Answer: False. Counterexample: $f(x) = x$ is an odd-degree polynomial. The derivative $f'(x) = 1$ does not have any zero (that is, its graph never hits the x-axis).