Last updated on 2016/12/14 at 15:44:42. Let me know of typos.

1 Mostly topics and skills from Test 1

1.1 Test 1: slope, secant line, tangent line, limit, derivative

1. Let $F(t)$ be a function defined by $F(t) = t^2 + 3t - 1$.

   (a) Evaluate the *difference quotient*

   $$\frac{F(t+\Delta t) - F(t)}{\Delta t}.$$

   Answer: $3 + \Delta t + 2t$

   (b) What is the slope of the secant line of the graph of $F(t)$ over $[t, t+\Delta t]$
   Answer: $3 + \Delta t + 2t$

   (c) What is the slope of the tangent line of the graph of $F(t)$ at t?
   Answer: $\lim_{\Delta t \to 0} 3 + \Delta t + 2t = 3 + 2t$.

   (d) Compute $F'(t)$.
   Answer: $F'(t) = 3 + 2t$

   (e) Compute the *net change* in F over the interval $[-1, 4]$.
   Answer: $F(4) - F(-1) = 27 - (-3) = 30$.

   (f) Compute the *average rate of change* in F over the interval $[-1, 4]$.
   Answer: $\frac{F(4)-F(-1)}{4-(-1)} = \frac{27-(-3)}{5} = \frac{30}{5} = 6$.

   (g) Write an equation for the secant line which meets the graph of f at $x = -1$ and $x = 4$.
   Answer: $(y - 27) = 6(x - 4)$. There are many possible equations written this way which describe this same secant line.

   (h) Write an equation for the secant line which meets the graph of f at $x = a$ and $x = 4$, where a is a real number that is smaller than 4.
   Answer:

   $$(y - 27) = \frac{(27 - (a^2 + 3a - 1))}{4 - a}(x - 4).$$

   That is,

   $$(y - 27) = \frac{(28 - a^2 - 3a)}{4 - a}(x - 4).$$

   There are many possible equations written this way which describe this same secant line.

2. Let $f(x) = x^3 - 2$.

   (a) Evaluate $f(x + \Delta x)$ (your answer should be a simple formula involving variables x and Δx.
   Answer: $-2 + (\Delta x)^3 + 3(\Delta x)^2 x + 3(\Delta x)x^2 + x^3$

(b) Evaluate and simplify the following quotient as much as possible.

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Answer:

$$(\Delta x)^2+3(\Delta x)x+3x^2$$

(c) Compute $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.
Answer: $3x^2$.

(d) Use the previous answer to compute $f'(x)$.
Answer: $f'(x)=3x^2$.

3. Function g is defined by

$$g(x)=\frac{(x-4)^{999}+2x}{70}.$$

(a) $g(5)-g(4)$
Answer: $g(5)-g(4)=(11-8)/70=3/70$.

(b) What is the *average rate of change in* g over the interval $[4,5]$?
Answer:

$$\frac{g(5)-g(4)}{5-4}=\frac{3}{70}$$

(c) Write an equation for the secant line which meets the graph of g at $x=4$ and $x=5$.
Answer:

$$\big(y-g(4)\big)=\frac{3}{70}(x-4).$$

That is,

$$\left(y-\frac{8}{70}\right)=\frac{3}{70}(x-4)$$

(d) $g(x+4)$.
Answer: Put $x+4$ into the boxes!

$$g(x+4)=\frac{\big((x+4)-4\big)^{999}+2(x+4)}{70}=\frac{x^{999}+2x+8}{70}$$

1.2 Test 1: Some Concepts from Test 1

1. True or False: If the average rate of change in a function f over $[a,b]$ is positive, then f must be increasing over $[a,b]$.
Answer: False. Counterexample: Consider $f(x)=x^2$ over the interval $[-1,3]$. The average rate of change is positive, but f is not increasing over this interval. (Warning: using our definition of *increasing*, f is also not decreasing and also not constant over this interval).

2. T/F: If f is a linear function, then f has the same net change over every possible interval.
Answer: False. Counterexample: Consider $f(x)=x$ and the intervals $[0,1]$ and $[0,2]$. The net change in f over $[0,1]$ is 1, and the net change in f over $[0,2]$ is 2.

3. T/F: If f is a linear function, then f has the same average rate of change over every possible interval.
   Answer: True. Explanation: If f is a linear function, then it has the form $f(x) = a_1x + a_0$, where a_1 and a_0 are real numbers. For any interval $[a, b]$, the average rate of change over $[a, b]$ is, by definition,
   $$\frac{f(b) - f(a)}{b - a} = \frac{a_1b + a_0 - (a_1a + a_0)}{b - a} = \frac{a_1b - a_1a}{b - a} = a_1.$$
4. Which of the following functions has/ have a constant rate of change (the same average rate of change over every possible interval)?
   a. $\frac{1}{9}x$ b. $\frac{9}{x}$ c. $\frac{1}{9x}$ d. $(\frac{1}{9}x - 1)(x + 9)$ e. $9x - \frac{1}{9}$
   Answer: Only a. $\frac{1}{9}x$ and e. $9x - \frac{1}{9}$ are of the form $a_1x + a_0$.

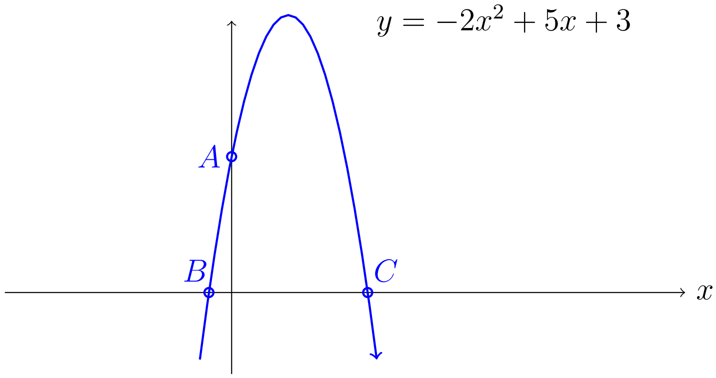
1.3 Test 1: Finding zeros of polynomials

1. Find the zeros of the following functions. If no zeros exist, say so.
   (a) $f(t) = t^{10} - 2t^9$.
   Answer: Set $0 = t^{10} - 2t^9$. But $t^{10} - 2t^9 = t^9(t - 2)$. So either $t^9 = 0$ or $t = 2$. So the zeros are 0 and 2.
   (b) $h(x) = (x - 2)(x^2 + 1)$.
   Answer: Set $0 = (x - 2)(x^2 + 1)$. So either $(x - 2) = 0$ or $(x^2 + 1) = 0$. There is no real number t such that $t^2 = -1$, hence there is only one zero, $x = 2$.
   (c) $g(x) = (x - 5)(x^3 + 1)$. (Note: this will involve long division).
   Answer: By the zero-factor theorem (p151), we know that $x = 5$ must be a zero of g.
   We also could see that $x = -1$ is a zero because $(-1)^3 + 1 = 0$. By the zero-factor theorem, we know that $(x + 1)$ must be a factor of $(x^3 + 1)$. Hence we can divide $(x^3 + 1)$ evenly by $(x + 1)$. We found that $(x^3 + 1) = (x + 1)(x^2 - x + 1)$. To find the other zeros of $(x^2 - x + 1)$, we set
   $$0 = x^2 - x + 1.$$
   Using the quadratic equation, we see that there are no real number solution. Therefore $(x^2 - x + 1)$ has no real roots (aka. real zeros). Hence g has exactly two zeros, $x = 5$ and $x = -1$.
   (d) $j(x) = (x + 5)^3(x^2 - 2)(x - \frac{1}{2})$.
   Answer: Note that $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, so $j(x) = (x + 5)^3(x - \sqrt{2})(x + \sqrt{2})(x - \frac{1}{2})$.
   Answer. Possible answer 1: By the zero-factor theorem, since $(x + 5)$ is a factor of j, -5 must be a zero of $j(x)$. Since $(x - \frac{1}{2})$ is a factor of $j(x)$, we know that $1/2$ must be a zero of j. Since $(x - \sqrt{2})$ and $(x + \sqrt{2})$ are factors of $j(x)$, we see that $x = -\sqrt{2}$ and $x = \sqrt{2}$ are zeros of j.
   Possible answer 2: Set $j(x)$ to 0. Then
   $$0 = (x + 5)^3 \left(x - \sqrt{2}\right) \left(x + \sqrt{2}\right) \left(x - \frac{1}{2}\right).$$
   Then either $0 = (x + 5)$ or $0 = \left(x - \sqrt{2}\right)$ or $0 = \left(x + \sqrt{2}\right)$ or $0 = \left(x - \frac{1}{2}\right)$. Then either $x = -5, \sqrt{2}, -\sqrt{2}$, or $x = \frac{1}{2}$.

(e) $h(x) = x^2 + x + 3$.
Answer: No zeros. Explanation: Set $0 = h(x)$. Using the quadratic formula, you can check that there are no real number solution.

1.4 Test 1: More Skills from Test 1

The figure shows the graph of a quadratic function:



1. Find the *exact* coordinates of each of the three marked points A, B, and C. Note, A is the point where the graph crosses the y-axis; B and C are the points where it crosses the x-axis.
   Answer: A $= (0, 3)$, $B = \left(-\frac{1}{2}, 0\right)$, $C = (3, 0)$.

2. Write down exactly all the *zero/s* (aka *root/s*) of this function.
   Answer: $x = -1/2$ and $x = 3$.

3. What degree is this polynomial? What is the leading term of this polynomial?
   Answer: degree 2, the leading term is $-2x^2$.

4. How many relative max or relative min does this graph has?
   Answer: 1. Explanation: there is at least one turning point because it goes toward $+\infty$ on the right, and toward $-\infty$ on the left. Since the maximal number of turning points that a polynomial of degree n is $n - 1$, there cannot be more than one turning point.

5. Write an equation of the secant line between the points $(0, 3)$ and $(2, 5)$.
   Answer:
   $$(y - 3) = 1(x - 0)$$

6. Write an equation of the secant line between the points $(1, 6)$ and $(2, 5)$.
   Answer:
   $$(y - 5) = -(x - 2)$$

1.5 Test 1: End behaviors and zeros of polynomials

1. Give an example of a polynomial of degree 7 which falls to the left, rises to the right, and has x=2, x=5, and x=11 as zeros. You do not need to sketch the graph, but you need to write the formula for the polynomial.
   Answer: $f(x) = (x - 2)(x - 5)(x - 11)x^4$

2. Give an example of a polynomial of degree 5 which rises to the left, falls to the right, and has x=2 as a zero. You do not need to sketch the graph, but you need to write the formula for the polynomial.
   Answer: $f(x) = (2 - x)^5$

3. Give an example of a polynomial of degree 6 which falls to the left, falls to the right, and has x=2 as a zero. You do not need to sketch the graph, but you need to write the formula for the polynomial.
   Answer: $f(x) = -(x - 2)^6$

4. Give an example of a polynomial which has $x = 2$ as a zero but does *not* change sign at $x = 2$.
   Answer: $(x - 2)^2$

5. Give an example of a polynomial which has $x = 2$ as a zero and changes sign from positive to negative at $x = 2$.
   Answer: $-(x - 2)$

6. True or False: every polynomial of even degree (2 or higher) has at least a zero. (If you write True, give an explanation. If write False, give a counterexample. If you give a counterexample, you need to give the formula for the polynomial.)
   Answer: False. Counterexample: $x^2 + 1$.

7. True or False: the polynomial $f(x) = x^2 - 4$ has exactly two zeros,
   Answer: True, at $x = -2$ and $x = 2$.

8. True or False: the polynomial $f(x) = x^2 - 4$ has a vertical asymptote at $x = -2$.
   Answer: False. A polynomial is defined at all real numbers, so it has no vertical asymptote.

9. True or False: every polynomial of odd degree has at least one zero.
   Answer: True. Explanation: A polynomial of odd degree has the property that either it falls to the left and rises to the right, or it rises to the left and falls to the right. Since a polynomial is continuous, it must cross the horizontal axis at least once.

2 From Test 2

2.1 Test 2: Evaluating limits

1. Let $F(t)$ be a function defined by

$$F(t) = \frac{t^2 + 3t - 1}{5}.$$

   (a) Evaluate $\lim_{t \to 0} F(t)$
   Answer:

$$\begin{aligned}\lim_{t \to 0} F(t) &= F(0) \text{ by direct substitution} \\ &= \frac{0^2 + 3(0) - 1}{5} \\ &= \frac{-1}{5}\end{aligned}$$

(b) Evaluate $\lim_{t\to 1} F(t)$
Answer:

$$\begin{aligned}\lim_{t\to 1} F(t) &= F(1) \text{ by direct substitution}\\ &= \frac{1^2+3(1)-1}{5}\\ &= \frac{3}{5}\end{aligned}$$

(c) Identify any points at which F is discontinuous.
Answer: None, F is continuous over $(-\infty,+\infty)$.

2. Evaluate $\lim_{t\to 0} 7+x-\sqrt{7}+\dfrac{x+2}{x+1}$.
Answer: $7-\sqrt{7}+2=9-\sqrt{7}$

3. Evaluate $\lim_{t\to 0^+} \dfrac{\sqrt{7+x}-\sqrt{7-x}}{2x}$.
Answer: $\frac{1}{2\sqrt{7}}$

4. Evaluate $\lim_{t\to 1^-} \dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}$.
Answer: This is from p247 #9. Multiply both top and bottom by the conjugate $\sqrt{2x+1}+\sqrt{3}$. Your final answer should be $\frac{1}{\sqrt{3}}$.

2.2 Test 2: Identifying Vertical asymptotes analytically

Identify the vertical asymptotes (if any) of following functions. If the function has no vertical asymptotes, say so.

1.
$$f(t)=\frac{5}{t^{10}-2t^9}.$$

Answer: First, we identify the points at which f is not continuous. To do this, we solve $0=t^{10}-2t^9$. But $t^{10}-2t^9=t^9(t-2)$. So either $t^9=0$ or $t=2$. So the function is discontinuous at $t=0$ and $t=2$.
In order for $t=c$ to be a vertical asymptote of $f(t)$, we must have either $\lim_{t\to c} f(t)=+\infty$ or $\lim_{t\to c} f(t)=-\infty$.
We check that the denominator $t^{10}-2t^9$ approaches 0 as t gets close to 0, and the numerator 5 approaches 5 as t gets close to 0. In class, we learned that this means that either either $\lim_{t\to 0} f(t)=+\infty$ or $\lim_{t\to 0} f(t)=-\infty$.
Similarly, we check that $\lim_{t\to 2} f(t)=+\infty$ or $\lim_{t\to 2} f(t)=-\infty$.
Hence, $t=0$ and $t=2$ are the vertical asymptotes of f.

2.
$$g(x)=\frac{x+1}{(x-5)(x^3+1)}.$$

Answer: To identify where g is not continuous, we look for the zeroes of the denominator $(x-5)(x^3+1)$. By the zero-factor theorem (p151), we know that $x=5$ must be a zero

of g.
We also could see that $x = -1$ is a zero because $(-1)^3 + 1 = 0$. By the zero-factor theorem, we know that $(x+1)$ must be a factor of (x^3+1). Hence we can divide (x^3+1) evenly by $(x+1)$. We found that $(x^3+1) = (x+1)(x^2-x+1)$. To find the other zeros of (x^2-x+1), we set

$$0 = x^2 - x + 1.$$

Using the quadratic equation, we see that there are no real number solution. Therefore (x^2-x+1) has no real roots (aka. real zeros). Hence the denominator of g has exactly two zeros, $x = 5$ and $x = -1$.

- First, we try to take the limit of g as x approaches -1.

$$\begin{aligned}\lim_{x\to-1}\frac{x+1}{(x-5)(x^3+1)} &= \lim_{x\to-1}\frac{x+1}{(x-5)(x+1)(x^2-x+1)} \text{ as I have computed above}\\ &= \lim_{x\to-1}\frac{1}{(x-5)(x^2-x+1)}\\ &= \frac{1}{(-1-5)((-1)^2-(-1)+1)} \text{ using direct substitution}\\ &= \frac{1}{6}.\end{aligned}$$

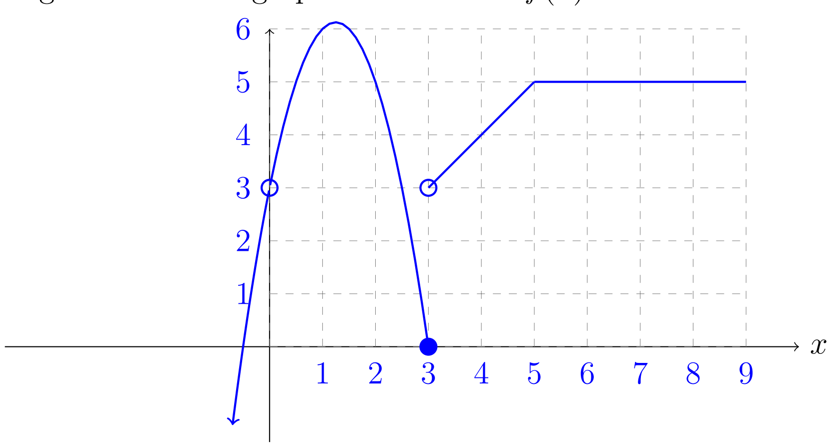
  Since $\lim_{x\to-1}$ exists, we know that $x = \frac{1}{6}$ is not a vertical asymptote of g. On the contrary, g has a removable discontinuity at $x = \frac{1}{6}$.

- Next, we try to take the limit of g as x approaches 5.
  As we try to do so, we notice that $\lim_{x\to5} x+1 = 6 \neq 0$, and $\lim_{x\to5}(x-5)(x^3+1) = (5-5)(5^3+1) = 0$. This was the last case that we learned in class, and we learned that $x = 5$ must be a vertical asymptote of g.

3. $h(x) = x^2 - 25x^2 + x + 3$.
   Answer: This function has no vertical asymptote. In fact, this function is continuous everywhere. Why?

2.3 Test 2: Reading and sketching graphs

The figure shows the graph of a function $f(x)$.



Read the value graph; if it does not exist or is not defined, just say so.

1. $\lim_{x\to 0^-} f(x)$
   Answer: 3

2. $\lim_{x\to 0^+} f(x)$
   Answer: 3

3. $\lim_{x\to 0} f(x)$
   Answer: 3

4. $\lim_{x\to 3^-} f(x)$
   Answer: 0

5. $\lim_{x\to 3^+} f(x)$
   Answer: 3

6. $\lim_{x\to 3} f(x)$
   Answer: The limit does not exist

7. $\lim_{x\to 4^-} f(x)$
   Answer: 4

8. $\lim_{x\to 4^+} f(x)$
   Answer: 4

9. $\lim_{x\to 4} \dfrac{f(x)}{11}$
   Answer: $\frac{4}{11}$

10. $\lim_{x\to 5^-} f(x)$
    Answer: 5

11. $\lim_{x\to 5^+} f(x)$
    Answer: 5

12. $\lim_{x\to 5} \dfrac{f(x)}{x}$
    Answer: 5

13. $\lim_{x\to 7^-} f(x)$
    Answer: 5

14. $\lim_{x\to 7^+} f(x)$
    Answer: 5

15. $\lim_{x\to 7} \dfrac{x+1}{f(x)}$
    Answer: 5

16. $f(0)$
    Answer: f is not defined at 0.

17. $f(3)$
    Answer: 0

18. $f(4)$
    Answer: 4

19. $f(5)$
    Answer: 5

20. $f(6)$
    Answer: 5

21. $f(7)$
    Answer: 5

22. For each part, answer True / False

    (a) f is continuous on [0,2].
    Answer: False

    (b) f is continuous on (0,2).
    Answer: True

    (c) f is continuous on [1,3].
    Answer: True

    (d) f is continuous on (1,3).
    Answer: True

    (e) f is continuous on [3,5].
    Answer: False

    (f) f is continuous on (3,5).
    Answer: True

    (g) f is continuous on [3,6].
    Answer: False

    (h) f is continuous on (3,6).
    Answer: True

    (i) f is continuous on [4,6].
    Answer: True

    (j) f is continuous on (4,6).
    Answer: True

23. For each part, answer True / False

    (a) $\lim_{x\to 0^-} f(x) = f(0)$.
    Answer: False

    (b) f is continuous at 0.
    Answer: True

    (c) f has a removable discontinuity at 0.
    Answer: True

    (d) f has a non-removable discontinuity at 0.
    Answer: False

    (e) $\lim_{x\to 2^+} f(x) = f(2)$.
    Answer: True

(f) f is continuous at 2.
Answer: True

(g) f has a removable discontinuity at 2.
Answer: False

(h) f has a non-removable discontinuity at 2.
Answer: False

(i) $\lim_{x\to 3^-} f(x) = f(3)$.
Answer: True

(j) f is continuous at 3.
Answer: False

(k) f has a removable discontinuity at 3.
Answer: False

(l) f has a non-removable discontinuity at 3.
Answer: True

(m) $\lim_{x\to 5} f(x) = f(5)$.
Answer: True

(n) f is continuous at 5.
Answer: True

(o) f has a removable discontinuity at 5.
Answer: False

(p) f has a non-removable discontinuity at 5.
Answer: False

(q) f is continuous on (1,3).
Answer: True

(r) f is continuous on [3,5].
Answer: False

(s) f is continuous on (3,5).
Answer: True

(t) f is continuous on [3,6].
Answer: False

(u) f is continuous on (3,6).
Answer: True

(v) f is continuous on [4,6].
Answer: True

(w) f is continuous on (4,6).
Answer: True

24. For each part, set up suitable axes and sketch the graph of a function satisfying all the given conditions. Make your drawing as clear as possible. If I cannot tell whether it's correct, I'll assume it's not.

(a) f(1)=2; f has a removable discontinuity at 1, but f is continuous everywhere else.

(b) f(1) is not defined; f has a removable discontinuity at 1, but f is continuous everywhere else.

(c) f(1)=2; f has a non-removable discontinuity at 1, but f is continuous everywhere else.

(d) f(1) is not defined; f has a non-removable discontinuity at 1, but f is continuous everywhere else.

(e) f is continuous everywhere except at $x = 2$ and $x = 5$, and $\lim_{x\to 2} f(x) = +\infty$ and $\lim_{x\to 5} f(x) = -\infty$.

(f) f is continuous on $(2,5)$, and $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 5^-} f(x) = +\infty$.

(g) f is continuous on $(2,5)$, and $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 5^-} f(x) = -\infty$.

(h) f is continuous on $(2,5)$, and $\lim_{x\to 2^+} f(x) = -\infty$ and $\lim_{x\to 5^-} f(x) = -\infty$

2.4 Test 2: Intermediate Value Theorem

1. Fill each blank with either an English word or a mathematical symbol so that the following is a complete and accurate statement of the Intermediate Value Theorem.

   If a __________ $f(x)$ is __________ on the __________ interval $[a,b]$, and a number M is between ____ and ____, then there is at least one point c in ______ where _____ = ____.

2. For each part, answer either True or False. For all parts, assume that f is continuous on $[0,2]$, that $f(0) = -10$ and $f(2) = 10$.

   (a) There must be at least one value c in $(0,2)$ where $f(c) = 10$.

   (b) There must be at least one value c in $[0,2]$ where $f(c) = 10$.

   (c) There must be at least one value c in $[0,2]$ where $f(c) = -1$.

   (d) There must be at least one value c in $[0,2]$ where $f(c) = -20$.

   (e) $f(1)$ must be between -10 and 10.

2.5 Test 2: Evaluating limits (part 2)

Evaluate the following limits. Whenever appropriate, answer with $+\infty$ or $-\infty$.

1. $\lim_{x\to 2^+} \dfrac{x}{x-2}$.
   Answer: $\lim_{x\to 2^+} \frac{x}{x-2} = +\infty$

2. $\lim_{x\to 2} \dfrac{1-x}{(x-2)^2}$.

   Answer: $\lim_{x\to 2} \dfrac{1-x}{(x-2)^2} = -\infty$

3. $\lim_{x\to 5^-} \dfrac{5-x}{(x-2)^2}$.

   Answer: $\lim_{x\to 5^-} \dfrac{5-x}{(x-2)^2} = \dfrac{0}{9} = 0$.

4. $\lim_{x \to 5^+} \frac{5+x}{(x-2)^2}$.
   Answer: $\lim_{x \to 5^+} \frac{5+x}{(x-2)^2} = \frac{10}{9}$.

5. $\lim_{x \to -1^+} \frac{x^3+1}{x+1}$.
   Answer: $\lim_{x \to (-1)^+} \frac{x^3+1}{x+1} = \lim_{x \to (-1)^+} \frac{(x+1)(x^2-x+1)}{x+1} = \lim_{x \to (-1)^+} x^2 - x + 1 = 3.$

3 Test 3

3.1 Test 3: the derivative function as a limit

Given the following functions for $f(x)$, use the *definition* (limit in sec 4.1) of derivative to evaluate $f'(x)$.

1. $f(x) = \sqrt{x+5}$
   Answer: $\frac{1}{2\sqrt{x+5}}$

2. $f(x) = \sqrt{x+3}$
   Answer: $\frac{1}{2\sqrt{x+3}}$

3. $f(x) = \sqrt{x+1}$
   Answer: $\frac{1}{2\sqrt{x+1}}$

4. $f(x) = 2x^3 + 1$

3.2 Test 3: Tangent lines

Suppose that $f(x) = \frac{-7}{(2x-3)^2}$.

1. Compute $f'(x)$ using derivative rules.
   Answer: $f'(x) = \frac{28}{(2x-3)^3}$.

2. Find the slope of the tangent line to the graph of f at $x = 5$
   Answer: $\frac{28}{343}$ or $\frac{4}{49}$

3. Write an equation for the tangent line at $x = 5$.
   Answer: $(y - f(5)) = \frac{4}{49}(x-5)$ which is also $(y + \frac{1}{7}) = \frac{4}{49}(x-5)$

3.3 Test 3: Applying the power rule, constant multiple,product rule, and quotient rule. Plus graphical applications of derivatives.

Compute the derivative of each of the following functions. Use the derivative rules. Do not set up a limit for these.

1. How do you explain why the power rule holds? Look back to the Handout:*Expanding Powers of Binomials [PDF]* from October 31 (Monday) lecture.

2. $f(x) = 3x^{10}$
   Answer:$30x^9$ by power rule and constant multiple

3. $f(x) = 3x^{-10}$
   Answer:$-30x^{-11}$ by power rule and constant multiple

4. $f(x) = -\frac{1}{3}x^{-10}$
   Answer:$\frac{10}{3}x^{-11}$ by power rule and constant multiple

5. $f(x) = -\frac{1}{x^3}$
   Answer: $\frac{3}{x^4}$ by power rule and constant multiple

6. $f(x) = \frac{42}{x^{10}}$
   Answer: $\frac{-420}{x^{11}}$ by power rule and constant multiple

7. $f(x) = \sqrt[7]{x}$
   Answer: $\frac{1}{7}x^{-6/7}$ by power rule

8. $f(x) = \sqrt[7]{2}$
   Answer: 0 because $\sqrt[7]{2}$ is a constant.

9. $f(x) = \sqrt[7]{2}\,x$
   Answer: $\sqrt[7]{2}$ by power rule and constant multiple

10. $f(x) = \frac{5x-2}{x^2+1}$
    Answer: Use the quotient rule to get $\frac{-5x^2+4x+5}{(x^2+1)^2}$. Sec 4.3 Example 3.

11. a.) $f(x) = \frac{3-(1/x)}{x+5}$
    Answer: Use the quotient rule to get $f'(x) = \frac{-3x^2+2x+5}{(x^2+5x)^2}$. See Sec 4.3 Example 4.

    b.) After that, find the points (x-values) where the graph of the function has a horizontal tangent line.
    Answer: If there is a horizontal tangent line at x then $-3x^2 + 2x + 5 = 0$.
    But$-3x^2 + 2x + 5 = -(3x-5)(x+1)$, so either $x = -1$ or $x = \frac{5}{3}$.

    c.) Use your answer from above to find an equation of the tangent line to the graph $f(x) = \frac{3-(1/x)}{x+5}$ at the point $(-1, 1)$.
    Answer: $y = 1$.

3.4 Test 3: Higher derivatives

1. Compute the second derivative $\frac{d^2}{dx^2}(3x^{10} + x^2)$
   Answer: $\frac{d}{dx}(30x^9) = 270x^8 + 2$

2. Compute the third derivative $\frac{d^3}{dx^3}(3x^{10} + 5x^2)$
   Answer: $\frac{d^2}{dx^2}(30x^9) = \frac{d}{dx}(270x^8) = 2160x^7$

3.5 Test 3: Knowing derivative rules

1. Suppose f is a differentiable function with respect to x. Apply one of the derivative rules rules and write down the derivative of $5\,f(x)$.
   Answer: $5\,f'(x)$

2. Suppose f is a differentiable function with respect to x. Apply the product rule, and write down the derivative of $x^2\, f(x)$.
   Answer: $x^2\, f'(x) + f(x)\, 2x$

3. Suppose f is a differentiable function with respect to x. Apply the quotient rule, and write down the derivative of $\frac{f(x)}{2x^5}$.
   Answer: $\frac{(2x^5)f'(x) - f(x)(10x^4)}{4x^{10}}$

3.6 Test 3: Figuring out from a graph whether f is continuous, not continuous, and differentiable

1. Sketch an example of a graph that is differentiable everywhere except at $x = 1$ and $x = 8$.
   Answer: see p258 (sec 4.1) in textbook Figures 4.11-4.13

2. Answer True or False. If a graph has a vertical tangent line at $x = 2$, then this graph is differentiable at $x = 2$.
   Answer: False. See Example 7 p258 (sec 4.1).

3. Answer True or False. If a graph has a horizontal tangent line at $x = 2$, then this graph is differentiable at $x = 2$.
   Answer: True. The derivative exists at $x = 2$ (and is equal to 0).

4. Answer True of False. If f is differentiable at $x = 5$, then f is continuous at $x = 5$.
   Answer: True by Theorem 4.1

5. Answer True of False. If f is not continuous at $x = 3$, then f is not differentiable at $x = 3$.
   Answer: True by (the contrapositive of) Theorem 4.1

6. Answer True of False. If f is continuous at $x = 2$, then f is differentiable at $x = 2$.
   Answer: False. For example the function $f(x) = |x - 2|$ is continuous at $x = 2$, but the it is not differentiable at $x = 2$ (see Example 4 sec 4.1, p258).

7. Suppose f is a function. What does it mean for f to be differentiable at $x = c$?
   Answer: f is differentiable at $x = c$ if

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(c)}{\Delta x}$$

   exists.
   See Sec 4.1 p255 Definition of the derivative of a function.

8. What does it mean for f to be continuous at $x = c$? Write a precise definition.
   Answer: Three conditions need to be met. See definition of continuity on p227 in Sec 3.4 textbook.

3.7 Test 3: Mean Value Theorem Practice

1. Let $f(x)$ be a function that is differentiable on an interval $[2, 5]$, and $f(2) = f(5) = -1$.

a. True or false. There is some c between 2 and 5 such that the tangent line to the graph f at $x = c$ is horizontal.
   Answer: True, by Rolle's Theorem/Mean Value Theorem (sec 5.2)

b. True or false. There is some c between 2 and 5 such that the tangent line to the graph f at $x = c$ is vertical.
Answer: False. Counterexample, consider the constant function $f(x) = -1$.

c. True or false. There is some c between 2 and 5 such that the tangent line to the graph f at $x = c$ has slope -1.
Answer: False. Counterexample, consider the constant function $f(x) = -1$.

2. Suppose you left Saint Peter at 9am and drove to your friend's house 100 miles away. You arrived at 11am, so you completed the trip in two hours.

a. What is your *average speed* for the entire trip?
Answer: $50mph$ (because you drove 100 miles in 2 hours).

b. Sometime between 9 and 11am, did your speedometer ever read exactly 50 mph? In other words, was your *instantaneous speed* ever exactly 50 mph? Yes or no, and why?
Answer: yes, due to Mean Value Theorem (sec 5.2)

3. Suppose f is differentiable on the interval $[1, 7]$. You also know that $f(1) = -3$, $f(6) = 5$, and $f(7) = 15$. Answer True or False. (If false, give a counterexample. If true, explain).

a. _____ f must also be continuous on the interval $[1, 7]$.
Answer: True, by Theorem 4.1 (textbook)

b. _____ There must be at least one number c in $(1, 7)$ where $f'(c) = 3$.
Answer: True, due to MVT, since $\frac{f(7)-f(1)}{7-1} = 3$

c. _____ There must be at least one number c in $(6, 7)$ where $f'(c) = 0$.
Answer: False. You can sketch a differentiable function that is increasing on $(6, 7)$. There is no point between 6 and 7 where the graph of this function has a horizontal tangent line.

3.8 Test 3: Functions and their derivatives

Answer True or False (If false, give a counterexample. If true, explain):

1. T/F: The derivative of a polynomial must be a polynomial.
Answer: True, due to the power rule, sum rule, and constant rule for derivatives.

2. T/F: The derivative of a rational function must be a rational function.
Answer: True. Explanation: Suppose $r(x)$ is a rational function. That is, by definition of rational functions, $r(x) = f(x)/g(x)$ for some polynomials $f(x)$, $g(x)$ where $g(x) \neq 0$. By the quotient rule, we know that $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$. We know that $f'(x)$ and $g'(x)$ are both polynomials because derivatives of polynomials are polynomials (see the previous question), so $g(x)f'(x) - f(x)g'(x)$ is a polynomial. Therefore, $\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$ is a polynomial.

3. T/F: The derivative of an even-degree polynomial must also be an even-degree polynomial.
Answer: False. Counterexample: The derivative of x^2 with respect to x is $2x$, which is an odd-degree polynomial.

4. T/F: The derivative of an odd-degree polynomial must also be an odd-degree polynomial.
Answer: False. Counterexample: The derivative of x^3 with respect to x is $3x^2$, which is an even-degree polynomial.

5. T/F: The derivative of an odd-degree polynomial must be an even-degree polynomial.
   Answer: True, because $\frac{d}{dx}(x^{2k+1}) = (2k+1)x^{2k}$, and $2k$ is even.

6. T/F: The derivative of an even-degree polynomial $f(x)$ (where the degree is 2 or higher) must have at least one zero).
   Answer: True. Explanation: The derivative of $f(x)$ must be a polynomial of odd degree. We learned about the end behaviors of an odd-degree polynomial (during the first month of the semester). It either falls to the left and rises to the right, or it rises to the left and falls to the right.

7. T/F: The derivative of an odd-degree polynomial $f(x)$ must have at least one zero.
   Answer: False. Counterexample: $f(x) = x$ is an odd-degree polynomial. The derivative $f'(x) = 1$ does not have any zero (that is, its graph never hits the x-axis).

3.9 From Test 4: Sec 4.4 Composition of functions, Chain Rule (plus all the previous derivative rules from Chapter 4)

1. Write $h(x) = \frac{1}{x+1}$ as a composition of two simpler functions. Then compute $h'(x)$ using Chain Rule.
   Ans: See Sec 4.4 Example 2a p. 286

2. Write $h(x) = \sqrt{3x^2 - x + 1}$ as a composition of two simpler functions. Then compute $h'(x)$ using Chain Rule
   Ans: See Sec 4.4 Example 2b p. 286

3. Use derivative rules to find $f'(x)$. On the test, do not spend a lot of time simplifying once the differentiation steps are completed.

   (a) $f(x) = (x^2 + 1)^3$. Check your answer with WolframAlpha.
   Ans: See Sec 4.4 Example 3 p. 287

   (b) $f(x) = \frac{-7}{(2x-3)^2}$. Check your answer with WolframAlpha.
   Ans: See Sec 4.4 Example 6 p. 287

   (c) $f(x) = \sqrt[3]{(x^2 - 1)^2}$. Check your answer with WolframAlpha.
   Ans: See Sec 4.4 Example 5 p. 287

4. Use your answer from above to find the critical numbers of $f(x) = \sqrt[3]{(x^2 - 1)^2}$. For definition of critical numbers, see Sec 5.1.
   Ans: The critical numbers of f are $x = -1$, $x = 0$, and $x = 1$. See Sec 4.4 Example 5 p. 287.

5. Read all of Sec 4.4, p284-289. Work through Examples 2-6 (chain rule). Work through Examples 6-9 (how to simplify after differentiation in a clear, readable way).

6. Book Exercises: Sec 4.4 #1-12 (odd numbers first); 47, 59, 60, 61.

3.10 From Test 4: Sec 4.5 Implicit Differentiation

1. Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$. (Hint: Differentiate both sides with respect to x, then solve for dy/dx).
   Ans: See Sec. 4.5 Example 2 p. 293.

2. Find the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $(\sqrt{2}, \frac{-1}{\sqrt{2}})$. (Hint: Differentiate both sides with respect to x, then solve for dy/dx, then get the slope you need).
   Ans: $\frac{1}{2}$. See Sec 4.5 Example 4 p. 294.

3. Find the equation of the tangent line to the graph of $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$. (Hint: Differentiate both sides with respect to x, then solve for dy/dx, then get the slope you need, then write the equation of the line).
   Ans: $y = 3x - \sqrt{2}$. See Sec 4.5 Example 8 p. 296.

4. Read Sec 4.5 p292-296 for many patiently-worked out implicit diff examples; you'll need to work through them yourself to get fluent with the process.

5. Book Exercises: After you do the reading and work the examples in the section, try Sec 4.5 p297 #1,4, 5, 15, 21.

4 From Test 4

4.1 From Test 4: Sec 4.6 Related Rates

1. Suppose that x and y are both differentiable functions of t, and they are related by the equation $y - x^2 = 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.
   Ans: 4. See Sec. 4.6 Example 1 on p. 300.

2. I have a 10-ft ladder that is leaning against a wall. At this particular moment, the distance between the bottom of the ladder and the wall is 8 feet. The bottom of the ladder is sliding away from the wall at the rate of 4 ft/second. Hence the top of the ladder is sliding down along the wall. Compute the rate of which the top of the ladder is sliding down at a particular time t. (Hint: See the Falling ladder related rates from Khan Academy).
   Ans: The top of the ladder is sliding down at 16/3 feet per second.

3. Read Sec 4.6 p300-302. Read the exposition and work through Examples #1-3.

4. Book Exercises: Sec 4.5 #25-31 (odd), 33a, 47; Sec 4.6 #15 (Hint: see Example 2 in the section).

5. Read book Sec 4.6 Ex. 4 (p301).

6. Rework Examples 1-5 (p300-303) on your own (solve it without looking at the book, and then compare to the textbook solution).

4.2 From Test 4: Sec 5.1 Extreme Value Theorem, Extrema

1. Give a complete and accurate statement of the Extreme Value Theorem.
   Ans: See Thm 5.1 p. 314.

2. Complete the definition. Let f be a function which is defined on c. Then we say that c is a *critical number* of f if ....
   Ans: See Sec 5.1 p. 316 (def of critical number).

3. True or False: If 5 is a critical number of f, then f has a relative minimum or relative maximum at $x = 5$.
   Answer: False. Counterexample: $f(x) = (x-5)^3$.

4. True or False: If f has a relative maximum at $x = 2$, then 2 is a critical number of f.
   Answer: True, by Thm 5.2 p. 316 from Sec 5.1.

5. Find the absolute minimum and absolute maximum of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.
   Ans: -1 and 16. See Sec 5.1 Example 2 on p. 317.

6. Answer True or False.

   (a) The maximum of a function that is continuous on a closed interval can occur at only one value in the interval.
   Ans: False. Counterexample, the maximum of x^2 occurs twice (at $x = -3$ and $x = 3$) on the closed interval $[-3, 3]$.

   (b) If a function is continuous on $[-3, 3]$, then it must have a minimum on the interval.
   Ans: True, by Extreme Value Thm.

   (c) If $x = 5$ is a critical number of the polynomial $p(x)$, then $x = 5$ must also be a critical number of the polynomial $f(x) = p(x) + 7$.
   Ans: True, since $f'(5) = p'(5)$ if it is defined. If $p'(5)$ is not defined, then $f'(5)$ is also not defined.

   (d) Read Sec 5.1 p314-318. Work through Ex. 2 and 3 on p317-318 (meaning, after you have read through them, copy the problem out and rework it on your own to see if you've really understood and remembered the process).

   (e) Book Exercises: Sec 5.1 (p319) #1-13 - most of these are answered just by looking at a graph, without any calculating. But you'll need to understand the terms and definitions from the reading!

   (f) More Book Exercises: Sec 5.1 p319 #14,16, 17, 19, 21, 45, 46, 47-50 (many of these are graphical exercises that don't require a lot of calculation)

4.3 From Test 4: Sec 5.3 Increase/Decrease/Min/Max

1. Let $f(x) = x^3 - \frac{3}{2}x^2$.

   (a) What are the critical numbers of f?
   Ans: 0 and 1

   (b) Determine the intervals of increase and decrease for f.
   Ans: See Sec 5.3 Example 1 on p. 330.

   (c) Classify the critical numbers (relative max/ relative min/ neither).
   Ans: f has a local max at the point where $x = 0$, and f has a local min at the point where $x = 1$. See the explanation at the top of p. 331.

2. Let $f(x) = 2x^3 - 3x^2 - 36x + 14$.

   (a) What are the critical numbers of f?
   Ans: -2 and 3

(b) Determine the intervals of increase and decrease for f.
Ans: See Sec 5.3 Example 2 on p. 332.

(c) Classify the critical numbers (relative max/ relative min/ neither).
Ans: f has a local max at the point where $x = -2$, and f has a local min at the point where $x = 3$. See Sec 5.3 Example 2 on p. 332.

3. Find the relative maximum and minimum of

$$f(x) = \frac{x^4 + 1}{x^2}.$$

Ans: See Sec 5.3 Example 4 on p. 334.

4. Answer True of False

(a) The sum of two increasing functions is increasing.
Ans: True

(b) Every nth degree polynomial has $(n - 1)$ critical numbers.
Ans: False. For example, how many critical numbers does x^5 has?

(c) There is a relative maximum or minimum at each critical number.
Ans: False. For example, x^5 has $x = 0$ as a critical number, but x^5 does not have a relative maximum or minimum at $x = 0$.

5. Read: Sec 5.3 p329-334.

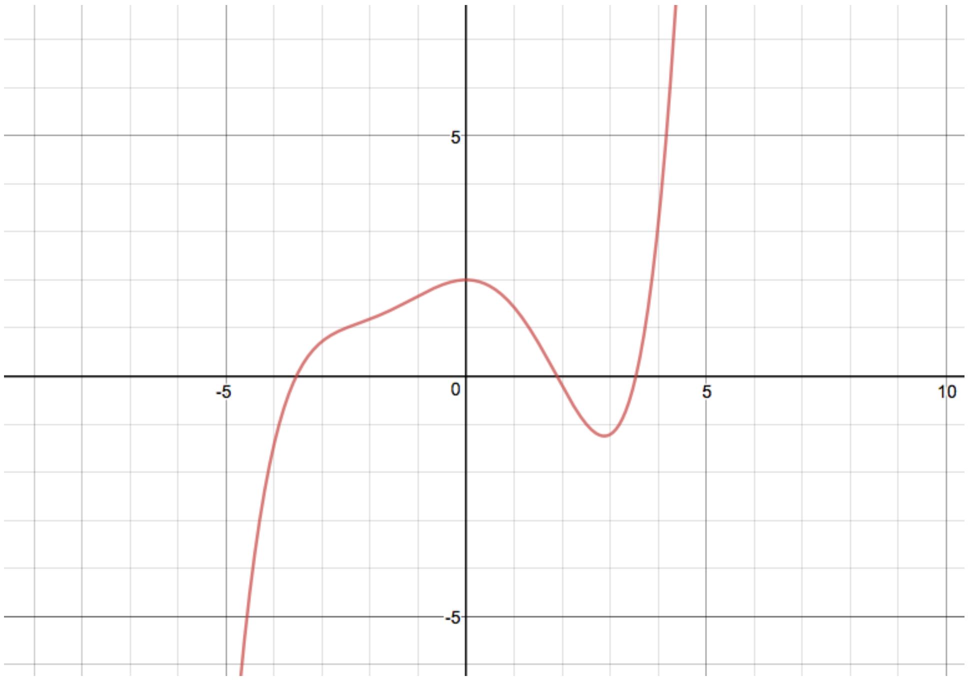
6. Memorize statements of Theorems 5.5 and 5.6.

7. Work through Examples 1-4 (read, understand, copy the problem, and rework on your own)

8. Book Exercises: Sec 5.3 p335 #1,3,8,11,15,17,19

4.4 From Test 4: Figuring out signs from graphs (Secs. 5.1,5.2,5.4)

The figure shows the graph of a function $f(x)$. Use the graph to determine the sign (positive, negative, or zero) of each of the following.



1. f(-4)
   Answer: negative
2. f(-3)
   Answer: positive
3. f(0)
   Answer: positive
4. f(1)
   Answer: positive
5. f(3)
   Answer: negative
6. $f'(-4)$
   Answer: pos
7. $f'(-3)$
   Answer: pos
8. $f'(0)$
   Answer: 0
9. $f'(1)$
   Answer: neg
10. $f'(4)$
    Answer: pos
11. $f''(-4)$
    Ans: neg
12. $f''(-3.5)$
    Ans: neg
13. $f''(0)$
    Ans: neg
14. $f''(1)$
    Ans: neg
15. $f''(3)$
    Ans: pos

4.5 From Test 4: Sec 5.4 Concavity

1. Complete the definition. Let f be differentiable on an open interval I.
   a. The graph of f is *concave up on* I means that ....
   b. The graph of f is *concave down on* I means that ....
   Ans: See defn in Sec 5.4 top of p. 338

2. Give a complete and accurate statement of the test for concavity (Thm 5.7 on p. 339).
   Ans: See Sec 5.4 Thm 5.7 Test for Concavity.

3. Determine the open intervals on which the graph of $\frac{6}{x^2+3}$ is concave up or down.
   Ans: Concave up on $(-\infty, -1)$ and $(1, \infty)$. Concave down on $(-1, 1)$. See Sec 5.4 Example 1 on p. 339.

4. Suppose f is a differentiable function on $(-\infty, \infty)$. Answer True or False.

   (a) If $f''(x)$ is positive on an interval (a, b), then $f(x)$ is concave up on (a, b).
   Ans: True, by Thm 5.7 Test for Concavity

(b) If $f''(x)$ is negative on an interval (a, b), then $f(x)$ is concave down on (a, b).
Ans: True, by Thm 5.7 Test for Concavity.

(c) If $f''(x)$ is positive on an interval (a, b), then $f(x)$ is increasing on (a, b).
Ans: False. Counterexample: Consider $f(x) = x^2$ on the interval $(-5, 5)$. Then $f''(x) = 2$, so $f''(x)$ is positive for all real x in $(-5, 5)$, but f is not increasing on $(-5, 5)$.

(d) If $f''(x)$ is negative on an interval (a, b), then $f(x)$ is decreasing on (a, b).
Ans: False. Counterexample: Consider $f(x) = -x^2$ on the interval $(-5, 5)$. Then $f''(x) = -2$, so $f''(x)$ is negative for all real x in $(-5, 5)$, but f is not decreasing on $(-5, 5)$.

5. If f is continuous everywhere, and f changes from being concave up to concave down at a point where $x = 5$, then this point (where $x = 5$) is a point of inflection of the graph f.
Ans: True, by definition of inflection point. See Sec 5.4 on p. 340.

6. Suppose $f(x) = x^N$ where N is an odd positive number larger than 1.

   (a) Does f have any inflection point/s? How many?
   Ans: Just one, at $x = 0$.

   (b) Determine the intervals on which the graph of f is concave up or down.
   Ans: concave down on the interval $(-\infty, 0)$, and concave up on the interval $(0, \infty)$.

7. Suppose $g(x) = x^N$ where N is an even positive number.

   (a) Does g have any inflection point/s? How many?
   Ans: None

   (b) Determine the intervals on which the graph of g is concave up or down.
   Ans: concave up on $(-\infty, \infty)$.

8. Answer True or False. These are exercises # 65-68 from p. 345.

   (a) Consider a cubic polynomial $h(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. This cubic polynomial has precisely one point of inflection.
   Ans: True. Explanation: the first derivative of h is $3ax^2 + 2bx + c$, so the second derivative of h is $6ax + 2b$. But $6ax + 2b = 0$ if $x = -\frac{b}{3a}$; $6ax + 2b$ is positive whenever $x > -\frac{b}{3a}$; and $6ax + 2b$ is negative whenever $x < -\frac{b}{3a}$.

   (b) If $f'(c) > 0$, then f is concave upward at $x = c$.
   Ans: False.

   (c) If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.
   Ans: False. For example, if $f(x) = x$, the graph have no inflection point even though $f''(2) = 0$.

9. Consider a cubic polynomial $h(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. Determine the intervals on which the graph of h is concave up or down.
Ans: Concave down on $(-\infty, -\frac{b}{3a})$. Concave up on $(-\frac{b}{3a}, \infty)$. See the explanation above for the T/F question.

10. Read Sec 5.4 p338-341. Work the examples and study the figures that go along with them!

11. Book Exercises: Sec 5.4 #1,2,3 (solve these analytically using the second derivative, but look at the graph that's provided to see if your answer matches), #9, 11, 13, 15, 37 and 38, 65.

5 After Test 4

5.1 Optimization problems (Sec 5.7)

1. You want to create an open box (with 4 walls and a base, but no top) with a square base. The surface area has to be 108 square inches. What dimensions (length of the square base, and height) will produce a box with maximum volume?
   Answer: $6 \times 6 \times 3$ inches. See Sec 5.7 Example 1 p. 364.

2. A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inches. What should be the dimensions of the page be so that the least amount of paper is used?
   Answer: 9 (in height) and 6 (in width) inches. See Sec 5.7 Example 3 p. 366.

5.2 Finding a function given its derivative

1. Could you give me a function whose derivative is equal to 0?
   Answer: any constant function will do, for example, $f(x) = 6$

2. Could you give me a function whose derivative is equal to 5?
   Answer: any linear function with slope 5, for example, $5x + 2$

3. Could you give me a function whose derivative is equal to $2x$?
   Answer: x^2 will work, and $x^2 + 8$ will also work.

4. Could you give me a function whose derivative is equal to $3x^2$?
   Answer: x^3 will work, and $x^3 + 8$ will also work.