Last updated on 2016/10/17 at 13:04:02. Please inform me of mistakes.

0.1 Topic: Evaluating limits

1. Let $F(t)$ be a function defined by

$$F(t) = \frac{t^2 + 3t - 1}{5}.$$

   (a) Evaluate

$$\lim_{t \to 0} F(t)$$

   Answer:

$$\begin{aligned} \lim_{t \to 0} F(t) &= F(0) \text{ by direct substitution} \\ &= \frac{0^2 + 3(0) - 1}{5} \\ &= \frac{-1}{5} \end{aligned}$$

   (b) Evaluate $\lim_{t \to 1} F(t)$ Answer:

$$\begin{aligned} \lim_{t \to 1} F(t) &= F(1) \text{ by direct substitution} \\ &= \frac{1^2 + 3(1) - 1}{5} \\ &= \frac{3}{5} \end{aligned}$$

   (c) Identify any points at which F is discontinuous.
   Answer: None, F is continuous over $(-\infty, +\infty)$.

2. Evaluate $\lim_{t \to 0} 7 + x - \sqrt{7} + \frac{x+2}{x+1}$. Answer: $7 - \sqrt{7} + 2 = 9 - \sqrt{7}$

3. Evaluate $\lim_{t \to 0^+} \frac{\sqrt{7+x} - \sqrt{7-x}}{2x}$. Answer: $\frac{1}{2\sqrt{7}}$

4. Evaluate $\lim_{t \to 1^-} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1}$. Answer: This is from p247 #9. Multiply both top and bottom by the conjugate $\sqrt{2x+1} + \sqrt{3}$. Your final answer should be $\frac{1}{\sqrt{3}}$.

0.2 Topic: Identifying Vertical asymptotes analytically

Identify the vertical asymptotes (if any) of following functions. If the function has no vertical asymptotes, say so.

1.

$$f(t) = \frac{5}{t^{10} - 2t^9}.$$

Answer: First, we identify the points at which f is not continuous. To do this, we solve

$0 = t^{10} - 2t^9$. But $t^{10} - 2t^9 = t^9(t-2)$. So either $t^9 = 0$ or $t = 2$. So the function is discontinuous at $t = 0$ and $t = 2$.

In order for $t = c$ to be a vertical asymptote of $f(t)$, we must have either $\lim_{t\to c} f(t) = +\infty$ or $\lim_{t\to c} f(t) = -\infty$.

We check that the denominator $t^{10} - 2t^9$ approaches 0 as t gets close to 0, and the numerator 5 approaches 5 as t gets close to 0. In class, we learned that this means that either either $\lim_{t\to 0} f(t) = +\infty$ or $\lim_{t\to 0} f(t) = -\infty$.

Similarly, we check that $\lim_{t\to 2} f(t) = +\infty$ or $\lim_{t\to 2} f(t) = -\infty$.

Hence, $t = 0$ and $t = 2$ are the vertical asymptotes of f.

2.
$$g(x) = \frac{x+1}{(x-5)(x^3+1)}.$$

Answer: To identify where g is not continuous, we look for the zeroes of the denominator $(x-5)(x^3+1)$. By the zero-factor theorem (p151), we know that $x = 5$ must be a zero of g.
We also could see that $x = -1$ is a zero because $(-1)^3 + 1 = 0$. By the zero-factor theorem, we know that $(x+1)$ must be a factor of (x^3+1). Hence we can divide (x^3+1) evenly by $(x+1)$. We found that $(x^3+1) = (x+1)(x^2-x+1)$. To find the other zeros of (x^2-x+1), we set
$$0 = x^2 - x + 1.$$

Using the quadratic equation, we see that there are no real number solution. Therefore (x^2-x+1) has no real roots (aka. real zeros). Hence the denominator of g has exactly two zeros, $x = 5$ and $x = -1$.

- First, we try to take the limit of g as x approaches -1.

$$\begin{aligned}
\lim_{x\to -1} \frac{x+1}{(x-5)(x^3+1)} &= \lim_{x\to -1} \frac{x+1}{(x-5)(x+1)(x^2-x+1)} \text{ as I have computed above}\\
&= \lim_{x\to -1} \frac{1}{(x-5)(x^2-x+1)}\\
&= \frac{1}{(-1-5)((-1)^2-(-1)+1)} \text{ using direct substitution}\\
&= \frac{1}{6}.
\end{aligned}$$

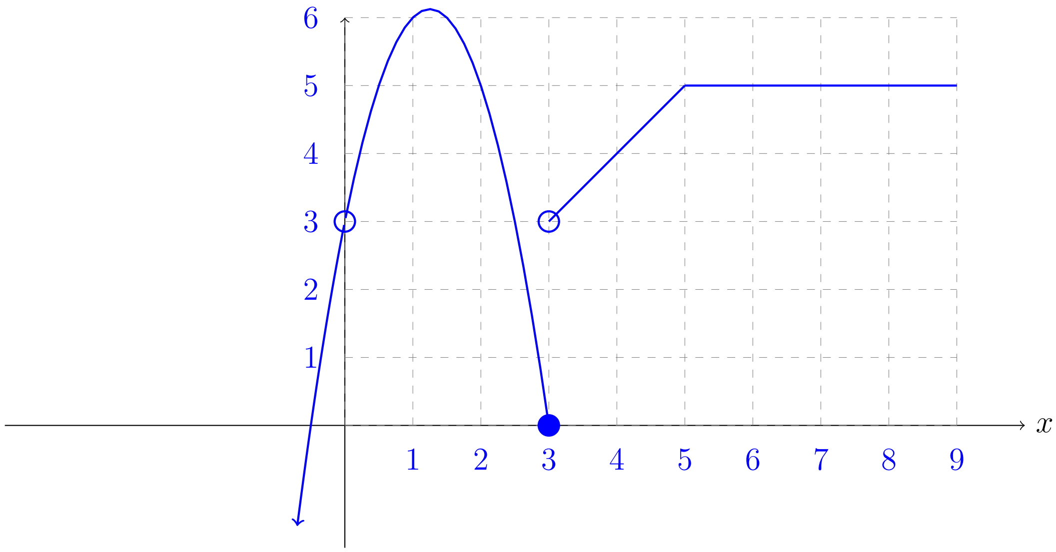
  Since $\lim_{x\to -1}$ exists, we know that $x = \frac{1}{6}$ is not a vertical asymptote of g. On the contrary, g has a removable discontinuity at $x = \frac{1}{6}$.

- Next, we try to take the limit of g as x approaches 5.
  As we try to do so, we notice that $\lim_{x\to 5} x + 1 = 6 \neq 0$, and $\lim_{x\to 5}(x-5)(x^3+1) = (5-5)(5^3+1) = 0$. This was the last case that we learned in class, and we learned that $x = 5$ must be a vertical asymptote of g.

3. $h(x) = x^2 - 25x^2 + x + 3$.
Answer: This function has no vertical asymptote. In fact, this function is continuous everywhere. Why?

0.3 Topic: Reading and sketching graphs

The figure shows the graph of a function $f(x)$.



Read the value graph; if it does not exist or is not defined, just say so.

1. $\lim_{x\to 0^-} f(x)$
   Answer: 3

2. $\lim_{x\to 0^+} f(x)$
   Answer: 3

3. $\lim_{x\to 0} f(x)$
   Answer: 3

4. $\lim_{x\to 3^-} f(x)$
   Answer: 0

5. $\lim_{x\to 3^+} f(x)$
   Answer: 3

6. $\lim_{x\to 3} f(x)$
   Answer: The limit does not exist

7. $\lim_{x\to 4^-} f(x)$
   Answer: 4

8. $\lim_{x\to 4^+} f(x)$
   Answer: 4

9. $\lim_{x\to 4} \dfrac{f(x)}{11}$
   Answer: $\frac{4}{11}$

10. $\lim_{x\to 5^-} f(x)$
    Answer: 5

11. $\lim_{x\to 5^+} f(x)$
    Answer: 5

12. $\lim_{x\to 5} \dfrac{f(x)}{x}$
    Answer: 5

13. $\lim_{x\to 7^-} f(x)$
    Answer: 5

14. $\lim_{x\to 7^+} f(x)$
    Answer: 5

15. $\lim_{x\to 7} \dfrac{x+1}{f(x)}$
    Answer: 5

16. $f(0)$
    Answer: f is not defined at 0.

17. $f(3)$
    Answer: 0

18. $f(4)$
    Answer: 4

19. $f(5)$
    Answer: 5

20. $f(6)$
    Answer: 5

21. $f(7)$
    Answer: 5

22. For each part, answer True / False

    (a) f is continuous on [0,2]. Answer: False
    (b) f is continuous on (0,2). Answer: True
    (c) f is continuous on [1,3]. Answer: True
    (d) f is continuous on (1,3). Answer: True
    (e) f is continuous on [3,5]. Answer: False
    (f) f is continuous on (3,5). Answer: True
    (g) f is continuous on [3,6]. Answer: False
    (h) f is continuous on (3,6). Answer: True
    (i) f is continuous on [4,6]. Answer: True
    (j) f is continuous on (4,6). Answer: True

23. For each part, answer True / False

(a) $\lim_{x\to 0^-} f(x) = f(0)$. Answer: False
(b) f is continuous at 0. Answer: True
(c) f has a removable discontinuity at 0. Answer: True
(d) f has a non-removable discontinuity at 0. Answer: False
(e) $\lim_{x\to 2^+} f(x) = f(2)$. Answer: True
(f) f is continuous at 2. Answer: True
(g) f has a removable discontinuity at 2. Answer: False
(h) f has a non-removable discontinuity at 2. Answer: False
(i) $\lim_{x\to 3^-} f(x) = f(3)$. Answer: True
(j) f is continuous at 3. Answer: False
(k) f has a removable discontinuity at 3. Answer: False
(l) f has a non-removable discontinuity at 3. Answer: True
(m) $\lim_{x\to 5} f(x) = f(5)$. Answer: True
(n) f is continuous at 5. Answer: True
(o) f has a removable discontinuity at 5. Answer: False
(p) f has a non-removable discontinuity at 5. Answer: False
(q) f is continuous on (1,3). Answer: True
(r) f is continuous on [3,5]. Answer: False
(s) f is continuous on (3,5). Answer: True
(t) f is continuous on [3,6]. Answer: False
(u) f is continuous on (3,6). Answer: True
(v) f is continuous on [4,6]. Answer: True
(w) f is continuous on (4,6). Answer: True

24. For each part, set up suitable axes and sketch the graph of a function satisfying all the given conditions. Make your drawing as clear as possible. If I cannot tell whether it's correct, I'll assume it's not.

(a) f(1)=2; f has a removable discontinuity at 1, but f is continuous everywhere else.
(b) f(1) is not defined; f has a removable discontinuity at 1, but f is continuous everywhere else.
(c) f(1)=2; f has a non-removable discontinuity at 1, but f is continuous everywhere else.
(d) f(1) is not defined; f has a non-removable discontinuity at 1, but f is continuous everywhere else.
(e) f is continuous everywhere except at $x = 2$ and $x = 5$, and $\lim_{x\to 2} f(x) = +\infty$ and $\lim_{x\to 5} f(x) = -\infty$.
(f) f is continuous on $(2, 5)$, and $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 5^-} f(x) = +\infty$.
(g) f is continuous on $(2, 5)$, and $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 5^-} f(x) = -\infty$.
(h) f is continuous on $(2, 5)$, and $\lim_{x\to 2^+} f(x) = -\infty$ and $\lim_{x\to 5^-} f(x) = -\infty$

0.4 Topic: Intermediate Value Theorem

1. Fill each blank with either an English word or a mathematical symbol so that the following is a complete and accurate statement of the Intermediate Value Theorem.

   If a __________ $f(x)$ is __________ on the __________ interval $[a, b]$, and a number M is between ____ and ____, then there is at least one point c in ______ where _____ $=$ ____.

2. For each part, answer either True or False. For all parts, assume that f is continuous on $[0, 2]$, that $f(0) = -10$ and $f(2) = 10$.

   (a) There must be at least one value c in $(0, 2)$ where $f(c) = 10$. Answer: False. Counterexample: the linear function which passes through both $(0, f(0))$ and $(2, f(2))$.

   (b) There must be at least one value c in $[0, 2]$ where $f(c) = 10$. Answer: True. For example, $c = 2$ works.

   (c) There must be at least one value c in $[0, 2]$ where $f(c) = -1$. Answer: True. Reasoning: the Intermediate Value Theorem.

   (d) There must be at least one value c in $[0, 2]$ where $f(c) = -20$. Answer: False. Counterexample: the linear function which passes through both $(0, f(0))$ and $(2, f(2))$.

   (e) $f(1)$ must be between -10 and 10. Answer: False.

0.5 Topic: Evaluating limits (part 2)

Evaluate the following limits. Whenever appropriate, answer with $+\infty$ or $-\infty$.

1. $\displaystyle\lim_{x\to 2^+} \frac{x}{x-2}$. Answer: $\lim_{x\to 2^+} \frac{x}{x-2} = +\infty$

2. $\displaystyle\lim_{x\to 2} \frac{1-x}{(x-2)^2}$. Answer: $\displaystyle\lim_{x\to 2} \frac{1-x}{(x-2)^2} = -\infty$

3. $\displaystyle\lim_{x\to 5^-} \frac{5-x}{(x-2)^2}$. Answer: $\displaystyle\lim_{x\to 5^-} \frac{5-x}{(x-2)^2} = \frac{0}{9} = 0$.

4. $\displaystyle\lim_{x\to 5^+} \frac{5+x}{(x-2)^2}$. Answer: $\displaystyle\lim_{x\to 5^+} \frac{5+x}{(x-2)^2} = \frac{10}{9}$.

5. $\displaystyle\lim_{x\to -1^+} \frac{x^3+1}{x+1}$.

   Answer: $\displaystyle\lim_{x\to(-1)^+} \frac{x^3+1}{x+1} = \lim_{x\to(-1)^+} \frac{(x+1)(x^2-x+1)}{x+1} = \lim_{x\to(-1)^+} x^2 - x + 1 = 3.$