Last updated on 2016/09/23 at 13:29:37. Please inform me of mistakes.

1. Let $F(t)$ be a function defined by $F(t) = t^2 + 3t - 1$.

   (a) Evaluate the *difference quotient*
   $$\frac{F(t+\Delta t) - F(t)}{\Delta t}.$$
   Answer: $3 + \Delta t + 2t$

   (b) Compute the *net change* in F over the interval $[-1, 4]$.
   Answer: $F(4) - F(-1) = 27 - (-3) = 30$.

   (c) Compute the *average rate of change* in F over the interval $[-1, 4]$.
   Answer: $\frac{F(4)-F(-1)}{4-(-1)} = \frac{27-(-3)}{5} = \frac{30}{5} = 6$.

   (d) Write an equation for the secant line which meets the graph of f at $x = -1$ and $x = 4$.
   Answer: $(y - 27) = 6(x - 4)$. There are many possible equations written this way which describe this same secant line.

   (e) Write an equation for the secant line which meets the graph of f at $x = a$ and $x = 4$, where a is a real number that is smaller than 4.
   Answer:
   $$(y - 27) = \frac{(27 - (a^2 + 3a - 1))}{4 - a}(x - 4).$$
   That is,
   $$(y - 27) = \frac{(28 - a^2 - 3a)}{4 - a}(x - 4).$$
   There are many possible equations written this way which describe this same secant line.

   (f) Write an equation for the secant line which meets the graph of f at $x = a$ and $x = b$, where a and b are real numbers such that $a < b$. Answer:
   $$(y - f(a)) = \frac{(f(b) - f(a))}{b - a}(x - a).$$
   That is,
   $$\big(y - (a^2 + 3a - 1)\big) = \frac{\big((b^2 + 3b) - (a^2 + 3a)\big)}{b - a}\big(x - a\big).$$
   There are many possible equations written this way which describe this same secant line.

2. Function g is defined by
   $$g(x) = \frac{(x-4)^{999} + 2x}{70}.$$

   (a) $g(5) - g(4)$
   Answer: $g(5) - g(4) = (11 - 8)/70 = 3/70$.

(b) What is the *average rate of change in g* over the interval $[4, 5]$?
Answer:

$$\frac{g(5) - g(4)}{5 - 4} = \frac{3}{70}$$

(c) Write an equation for the secant line which meets the graph of g at $x = 4$ and $x = 5$.
Answer:

$$\big(y - g(4)\big) = \frac{3}{70}(x - 4).$$

That is,

$$\left(y - \frac{8}{70}\right) = \frac{3}{70}(x - 4)$$

(d) $g(x + 4)$.
Answer: Put $x + 4$ into the boxes!

$$g(x + 4) = \frac{\big((x + 4) - 4\big)^{999} + 2(x + 4)}{70} = \frac{x^{999} + 2x + 8}{70}$$

3. (a) Give an example of a polynomial of degree 7 which falls to the left, rises to the right, and has x=2, x=5, and x=11 as zeros. You do not need to sketch the graph, but you need to write the formula for the polynomial.
   Answer: $f(x) = (x-2)(x-5)(x-11)x^4$

   (b) Give an example of a polynomial of degree 5 which rises to the left, falls to the right, and has x=2 as a zero. You do not need to sketch the graph, but you need to write the formula for the polynomial.
   Answer: $f(x) = (2-x)^5$

   (c) Give an example of a polynomial of degree 6 which falls to the left, falls to the right, and has x=2 as a zero. You do not need to sketch the graph, but you need to write the formula for the polynomial.
   Answer: $f(x) = -(x-2)^6$

   (d) True or False: every polynomial of even degree (2 or higher) has at least a zero. (If you write True, give an explanation. If write False, give a counterexample. If you give a counterexample, you need to give the formula for the polynomial.)
   Answer: False. Counterexample: x^2+1.

   (e) True or False: every polynomial of odd degree has at least one zero. (If you write True, give an explanation. If write False, give a counterexample. If you give a counterexample, you need to write the formula for the polynomial.)
   Answer: True. Explanation: A polynomial of odd degree has the property that either it falls to the left and rises to the right, or it rises to the left and falls to the right. Since a polynomial is continuous, it must cross the horizontal axis at least once.

   (f) True or False: If the average rate of change in a function f over $[a, b]$ is positive, then f must be increasing over $[a, b]$.
   Answer: False. Counterexample: Consider $f(x) = x^2$ over the interval $[-1, 3]$. The average rate of change is positive, but f is not increasing over this interval. (Warning: using our definition of *increasing*, f is also not decreasing and also not constant over this interval).

   (g) True or False: If f is a linear function, then f has the same net change over every possible interval.
   Answer: False. Counterexample: Consider $f(x) = x$ and the intervals $[0, 1]$ and $[0, 2]$. The net change in f over $[0, 1]$ is 1, and the net change in f over $[0, 2]$ is 2.

   (h) True or False: If f is a linear function, then f has the same average rate of change over every possible interval.
   Answer: True. Explanation: If f is a linear function, then it has the form $f(x) = a_1x + a_0$, where a_1 and a_0 are real numbers. For any interval $[a, b]$, the average rate of change over $[a, b]$ is, by definition,

$$\frac{f(b)-f(a)}{b-a} = \frac{a_1b + a_0 - (a_1a + a_0)}{b-a} = \frac{a_1b - a_1a}{b-a} = a_1.$$

   (i) Which of the following functions has/ have a constant rate of change (the same average rate of change over every possible interval)?
   a. $\frac{1}{9}x$ b. $\frac{9}{x}$ c. $\frac{1}{9x}$ d. $(\frac{1}{9}x-1)(x+9)$ e. $9x-\frac{1}{9}$
   Answer: Only a. $\frac{1}{9}x$ and e. $9x-\frac{1}{9}$ are of the form $a_1x + a_0$.

4. Find the zeros of the following functions. If no zeros exist, say so.

   (a) $f(t) = t^{10} - 2t^9$.
   Solution: Set $0 = t^{10} - 2t^9$. But $t^{10} - 2t^9 = t^9(t-2)$. So either $t^9 = 0$ or $t = 2$. So the zeros are 0 and 2.

   (b) $h(x) = (x-2)(x^2+1)$. (Note: you saw this yesterday).
   Answer: Set $0 = (x-2)(x^2+1)$. So either $(x-2) = 0$ or $(x^2+1) = 0$. There is no real number t such that $t^2 = -1$, hence there is only one zero, $x = 2$.

   (c) $g(x) = (x-5)(x^3+1)$. (Note: you saw this yesterday. This may be a challenging problem).
   Answer: By the zero-factor theorem (p151), we know that $x = 5$ must be a zero of g.
   We also could see that $x = -1$ is a zero because $(-1)^3 + 1 = 0$. By the zero-factor theorem, we know that $(x+1)$ must be a factor of (x^3+1). Hence we can divide (x^3+1) evenly by $(x+1)$. We found that $(x^3+1) = (x+1)(x^2-x+1)$. To find the other zeros of (x^2-x+1), we set

   $$0 = x^2 - x + 1.$$

   Using the quadratic equation, we see that there are no real number solution. Therefore (x^2-x+1) has no real roots (aka. real zeros). Hence g has exactly two zeros, $x = 5$ and $x = -1$.

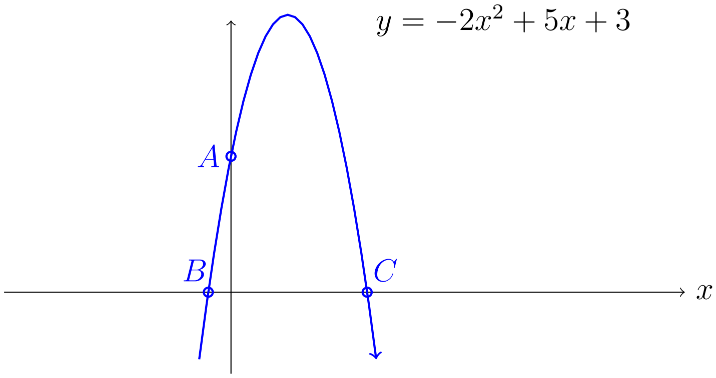
   (d) $j(x) = (x+5)^3(x^2-2)(x-\frac{1}{2})$.
   Answer: Note that $x^2 - 2 = (x-\sqrt{2})(x+\sqrt{2})$, so $j(x) = (x+5)^3(x-\sqrt{2})(x+\sqrt{2})(x-\frac{1}{2})$.
   Possible answer 1: By the zero-factor theorem, since $(x+5)$ is a factor of j, -5 must be a zero of $j(x)$. Since $(x-\frac{1}{2})$ is a factor of $j(x)$, we know that $1/2$ must be a zero of j. Since $(x-\sqrt{2})$ and $(x+\sqrt{2})$ are factors of $j(x)$, we see that $x = -\sqrt{2}$ and $x = \sqrt{2}$ are zeros of j.
   Possible answer 2: Set $j(x)$ to 0. Then

   $$0 = (x+5)^3\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)\left(x-\frac{1}{2}\right).$$

   Then either $0 = (x+5)$ or $0 = \left(x-\sqrt{2}\right)$ or $0 = \left(x+\sqrt{2}\right)$ or $0 = \left(x-\frac{1}{2}\right)$. Then either $x = -5, \sqrt{2}, -\sqrt{2}$, or $x = \frac{1}{2}$.

   (e) $h(x) = x^2 + x + 3$.
   Answer: No zeros. Explanation: Set $0 = h(x)$. Using the quadratic formula, you can check that there are no real number solution.

5. The figure shows the graph of a quadratic function:



   (a) Find the *exact* coordinates of each of the three marked points A, B, and C. Note, A is the point where the graph crosses the y-axis; B and C are the points where it crosses the x-axis.
   Answer: A $= (0, 3)$, $B = \left(-\frac{1}{2}, 0\right)$, $C = (3, 0)$.

   (b) Write down exactly all the *zero/s* (aka *root/s*) of this function.
   Answer: $x = -1/2$ and $x = 3$.

   (c) What degree is this polynomial? What is the leading term of this polynomial?
   Answer: degree 2, the leading term is $-2x^2$.

   (d) How many turning points does this graph has?
   Answer: 1. Explanation: there is at least one turning point because it goes toward $+\infty$ on the right, and toward $-\infty$ on the left. Since the maximal number of turning points that a polynomial of degree n is $n-1$, there cannot be more than one turning point.

   (e) Write an equation of the secant line between the points $(0, 3)$ and $(2, 5)$.
   Answer:
$$(y - 3) = 1(x - 0)$$

   (f) Write an equation of the secant line between the points $(1, 6)$ and $(2, 5)$.
   Answer:
$$(y - 5) = -(x - 2)$$

6. Let $f(x) = x^3 - 2$.

   (a) Evaluate $f(x+\Delta x)$ (your answer should be a simple formula involving two variables x and Δx.(your answer should be a simple formula involving two variables x and Δx.
   Answer: $-2 + (\Delta x)^3 + 3(\Delta x)^2 x + 3(\Delta x)x^2 + x^3$

   (b) Evaluate and simplify the following quotient as much as possible.
$$\frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

   Answer:
$$(\Delta x)^2 + 3(\Delta x)x + 3x^2$$