Expanding powers of (x+Δx)   (or any binomial)

If you organize the expansion of $(x + \Delta x)^n$ well, it's not too hard to proceed from one power to the next power. When we do this, we'll see some patterns emerge that are useful in computing derivatives of power functions.

Second Power

```
prettyMultiplication[2]
```

$(x + \Delta x)^2 == (x + \Delta x)\ (x + \Delta x)$

$$\begin{array}{lllllll} = & x^2 & + & x\,\Delta x & & \\ & & + & x\,\Delta x & + & \Delta x^2 \\ \hline = & x^2 & + & 2\,x\,\Delta x & + & \Delta x^2 \end{array}$$

Third Power

We get the third power of (x+Δx) by multiplying that previous result by another (x+Δx):

```
prettyMultiplication[3]
```

$(x + \Delta x)^3 == (x + \Delta x)^2\ (x + \Delta x)$

$$\begin{array}{lllllllll} = & x^3 & + & 2\,x^2\,\Delta x & + & x\,\Delta x^2 & & \\ & & + & x^2\,\Delta x & + & 2\,x\,\Delta x^2 & + & \Delta x^3 \\ \hline = & x^3 & + & 3\,x^2\,\Delta x & + & 3\,x\,\Delta x^2 & + & \Delta x^3 \end{array}$$

Fourth Power...

...multiply the previous result by another (x+Δx)

```
prettyMultiplication[4]
```

$(x + \Delta x)^4 == (x + \Delta x)^3\ (x + \Delta x)$

$$\begin{array}{lllllllllll} = & x^4 & + & 3\,x^3\,\Delta x & + & 3\,x^2\,\Delta x^2 & + & x\,\Delta x^3 & & \\ & & + & x^3\,\Delta x & + & 3\,x^2\,\Delta x^2 & + & 3\,x\,\Delta x^3 & + & \Delta x^4 \\ \hline = & x^4 & + & 4\,x^3\,\Delta x & + & 6\,x^2\,\Delta x^2 & + & 4\,x\,\Delta x^3 & + & \Delta x^4 \end{array}$$

Notice how the same set of coefficients shows up in each of the two lines that we're adding together each time. They're just shifted copies of the coefficients from the previous power.

...and so on

(multiply the result from $(x+\Delta x)^4$ by another factor of (x+Δx) and add like terms)

```
prettyMultiplication[5]
```

$$(x+\Delta x)^5 == (x+\Delta x)^4\,(x+\Delta x)$$

$$\begin{array}{cccccccccccc}
= & x^5 & + & 4\,x^4\,\Delta x & + & 6\,x^3\,\Delta x^2 & + & 4\,x^2\,\Delta x^3 & + & x\,\Delta x^4 & & \\
 & & + & x^4\,\Delta x & + & 4\,x^3\,\Delta x^2 & + & 6\,x^2\,\Delta x^3 & + & 4\,x\,\Delta x^4 & + & \Delta x^5 \\
\hline
= & x^5 & + & 5\,x^4\,\Delta x & + & 10\,x^3\,\Delta x^2 & + & 10\,x^2\,\Delta x^3 & + & 5\,x\,\Delta x^4 & + & \Delta x^5
\end{array}$$

(then multiply the result from $(x+\Delta x)^5$ by another factor of (x+Δx) and add like terms)

```
prettyMultiplication[6]
```

$$(x+\Delta x)^6 == (x+\Delta x)^5\,(x+\Delta x)$$

$$\begin{array}{cccccccccccccc}
= & x^6 & + & 5\,x^5\,\Delta x & + & 10\,x^4\,\Delta x^2 & + & 10\,x^3\,\Delta x^3 & + & 5\,x^2\,\Delta x^4 & + & x\,\Delta x^5 & & \\
 & & + & x^5\,\Delta x & + & 5\,x^4\,\Delta x^2 & + & 10\,x^3\,\Delta x^3 & + & 10\,x^2\,\Delta x^4 & + & 5\,x\,\Delta x^5 & + & \Delta x^6 \\
\hline
= & x^6 & + & 6\,x^5\,\Delta x & + & 15\,x^4\,\Delta x^2 & + & 20\,x^3\,\Delta x^3 & + & 15\,x^2\,\Delta x^4 & + & 6\,x\,\Delta x^5 & + & \Delta x^6
\end{array}$$

...and so on!

Table: Expansion of the first few powers of (x+Δx)

Power	Expansion
$(x+\Delta x)^2$	$x^2 + 2\,x\,\Delta x + \Delta x^2$
$(x+\Delta x)^3$	$x^3 + 3\,x^2\,\Delta x + 3\,x\,\Delta x^2 + \Delta x^3$
$(x+\Delta x)^4$	$x^4 + 4\,x^3\,\Delta x + 6\,x^2\,\Delta x^2 + 4\,x\,\Delta x^3 + \Delta x^4$
$(x+\Delta x)^5$	$x^5 + 5\,x^4\,\Delta x + 10\,x^3\,\Delta x^2 + 10\,x^2\,\Delta x^3 + 5\,x\,\Delta x^4 + \Delta x^5$
$(x+\Delta x)^6$	$x^6 + 6\,x^5\,\Delta x + 15\,x^4\,\Delta x^2 + 20\,x^3\,\Delta x^3 + 15\,x^2\,\Delta x^4 + 6\,x\,\Delta x^5 + \Delta x^6$
$(x+\Delta x)^7$	$x^7 + 7\,x^6\,\Delta x + 21\,x^5\,\Delta x^2 + 35\,x^4\,\Delta x^3 + 35\,x^3\,\Delta x^4 + 21\,x^2\,\Delta x^5 + 7\,x\,\Delta x^6 + \Delta x^7$
$(x+\Delta x)^8$	$x^8 + 8\,x^7\,\Delta x + 28\,x^6\,\Delta x^2 + 56\,x^5\,\Delta x^3 + 70\,x^4\,\Delta x^4 + 56\,x^3\,\Delta x^5 + 28\,x^2\,\Delta x^6 + 8\,x\,\Delta x^7 + \Delta x^8$

Exercise: Reproduce this table by hand, and practice until you can do it fluently.

Observation: The expansion of $(x+\Delta x)^n$ has the form (fill in the blanks)

________ + ________ + (stuff involving higher powers of Δx)