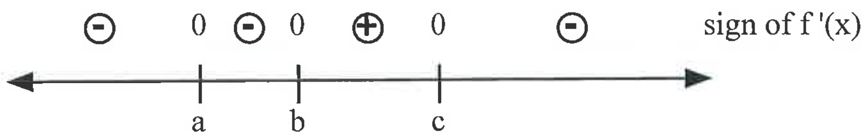
1. Assume that f is a differentiable function. The sign of $f'(x)$ is shown on the following chart.



a. Complete the following sentences (the words positive, negative, increasing, and/or decreasing are relevant).

$f'(x)$ is negative on $(-\infty, a)$, (a, b), and $(c, +\infty)$,

and $f'(x)$ is positive on (b, c).

this is what the chart is telling us directly about $f'(x)$

So (by Theorem 5.5),

$f(x)$ is decreasing on $(-\infty, a)$, (a, b), and $(c, +\infty)$

and $f(x)$ is increasing on (b, c).

this is what Theorem 5.5 tells us about the function $f(x)$.

b. How many critical points does f have? 3 (the points a, b, and c where f' is zero).

c. How many times does f' change sign (from positive to negative, or vice versa)?

only twice: from − to + at b
and from + to − at c

d. How many relative minima does f have?

1, at b.

at b, f looks like

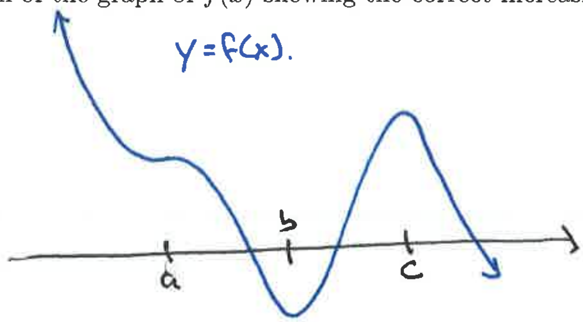
e. How many relative maxima does f have?

1, at c

at c, f looks like

(Theorem 5.6).

f. Make a rough sketch of the graph of $f(x)$ showing the correct increasing/decreasing behavior.



It must be something like this. We don't have enough info. to be sure about the exact y-values on the graph of f, only its increasing/decreasing behavior.

For the remaining questions, assume $f(x)$ is a polynomial.

g. Describe the end behavior of f as we would have done earlier in the semester: f is <u>rising</u> to the left and <u>Falling</u> to the right.

this is potentially confusing: f is "rising to the left" actually means f is <u>decreasing</u> on the leftmost part of the graph.

h. True/False: The function $f(x)$ must have at least one real zero. (Explain!)

If f is a polynomial that is rising to the left & Falling to the right,

then it must be an odd-degree polynomial, and it must have at least one real zero.

i. What's the smallest possible degree $f(x)$ could have?
(As always, explain!)

Well, $f'(x)$ has three zeros.

so $f'(x)$ must be at least degree three.

That means $f(x)$ must be at least degree four.

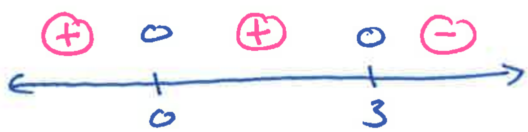
But the end behavior of f means it can't be degree 4 (even)

so actually $f(x)$ must be degree 5, at least!

2. Suppose $f'(x) = -2x^2(x-3)$.

a. Make a chart showing the critical numbers of f and the sign of f' on the remaining intervals.] these would make more sense in the other order!

b. How many critical numbers does f have? What are they?



sign of $f'(x)$

critical numbers: where $-2x^2(x-3) = 0$

either $x = 0$ or $x = 3$

c. How many times does f' change sign (from positive to negative, or vice versa)?

just once, at $x=3$.

d. How many relative minima does f have? (Where do they occur, if at all?)

no relative minima at all!

we never see f' change from − to +.

e. How many relative maxima does f have? (Where do they occur, if at all?)

1 relative max at $x=3$,

where f' changes from + to −

3. Let $f(x) = x^4 + 2x^2$. Compute $f'(x)$ and carry out parts (a)-(e) for this function as in #2.

$f'(x) = 4x^3 + 4x$

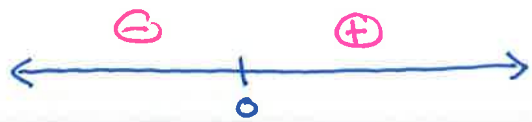
so critical numbers: where $4x^3 + 4x = 0$

$4x(x^2+1) = 0$

either $x = 0$ or $x^2 + 1 = 0$

but this has no real solutions.

So $x = 0$ is the only critical number.



sign of $f'(x) = 4 \cdot x \cdot (x^2+1)$

f has a relative min. at $x = 0$, where $f'(x)$ changes from $-$ to $+$ (f looks like: )

Increase/Decrease/Max/Min #1

Here's a polynomial f(x) whose derivative would match the pattern shown in the sign chart.
I have the computer find and factor f'(x) so you can see the critical points are at x=2, x=4 and x=8.

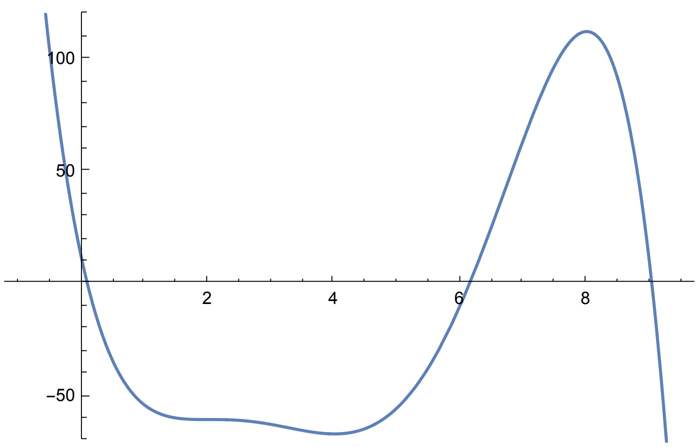
```
f[x_] := (-1 / 5) x^5 + 4 x^4 - 28 x^3 + 88 x^2 - 128 x + 10
f[x] // TraditionalForm
f'[x] // Factor // TraditionalForm
```

$$-\frac{x^5}{5} + 4x^4 - 28x^3 + 88x^2 - 128x + 10$$

$$-(x-8)(x-4)(x-2)^2$$

Notice that the factor which occurs with an even power (x=2) in f'(x) is the one that *doesn't* give a sign change in f'(x), and *doesn't* give a max or min on the graph of f(x). But f(x) has a relative min at x=4 and a relative max at x=8, as predicted:

```
Plot[f[x], {x, -1, 9.5}, PlotRange → {-70, 120}]
```



Increase/Decrease/Max/Min #2

Here’s a function f(x) that has the derivative given in #2 - I just use the computer to verify that the derivative is what I wanted it to be (notice the computer puts the factors in kind of a weird order):

```
f[x_] := (-1 / 2) x^4 + 2 x^3
f'[x]
Factor[f'[x]]
```

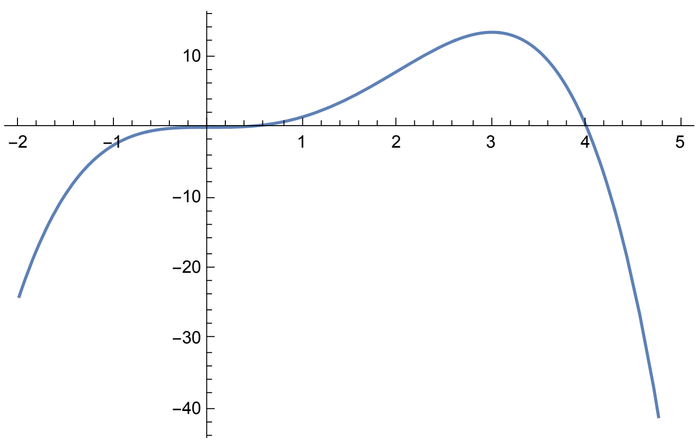
$6 x^2 - 2 x^3$

$-2 (-3 + x) x^2$

And here’s a plot of that function f(x) - notice there’s no max or min at x=0 (even though x=0 is a critical number). There’s just a pause: the tangent line is horizontal at x=0 but then f(x) resumes increasing.

At x=3, we see the expected relative maximum. Based on the increasing/decreasing behavior of f(x), you could actually have reasoned out that f(3) must actually be the absolute maximum value of f(x) - not merely a relative max, but the actual biggest output that f ever produces.

```
Plot[f[x], {x, -2, 5}]
```



Increase/Decrease/Max/Min #3

```
f[x_] := x^4 + 2 x^2
```

The plot of f(x) shows, as expected, just a minimum at x=0. We could have predicted the end behavior just from the degree and the leading coefficient, but calculus shows there isn't any other interesting behavior in between - it just has the one turning point that it needs in order to rise on both ends.

```
Plot[f[x], {x, -1, 1}]
```