Cluster algebras from surfaces

Lecture notes for the CIMPA School
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Chapter 0

Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [FZ1] in 2002. Their original motivation was coming from canonical bases in Lie Theory. Today cluster algebras are connected to various fields of mathematics, including

- Combinatorics (polyhedra, frieze patterns, green sequences, snake graphs, T-paths, dimer models, triangulations of surfaces)
- Representation theory of finite dimensional algebras (cluster categories, cluster-tilted algebras, preprojective algebras, tilting theory, 2-Calbi-Yau categories, invariant theory)
- Poisson geometry and algebraic geometry (cluster varieties, Grassmannians, stability conditions, scattering diagrams, Poisson structures on SL(n))
- Teichmüller theory (lambda-lengths, Penner coordinates, cluster varieties)
- Knot theory (Chern-Simons invariants, volume conjecture, Legendrian knots)
- Dynamical systems (frieze patterns, pentagram map, T-systems, sine-Gordon Y-systems)
- Mathematical Physics (statistical mechanics, Donaldson-Thomas invariants, quantum dilogarithm identities, BPS particles)

Furthermore, because of an intensive research over the last 15 years, the subject of cluster algebras itself has grown into an independent theory.

In this minicourse, we will focus on cluster algebras from surfaces, a special class of cluster algebras. The first chapter is a short introduction to cluster algebras, and chapters two, three and four are devoted to cluster algebras from surfaces, especially to the expansion formulas for the cluster variables and the construction of canonical bases in terms of snake and band graphs. [1]

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Chapter 1

Cluster algebras

The definition of cluster algebras is elementary, but quite complicated. We describe it in this first section. Since these notes are aiming for cluster algebras from surfaces, we do not present the most general definition of cluster algebras, but restrict ourselves to so-called skew-symmetric cluster algebras with principal coefficients. For the general definition and further details we refer to [FZ4].

1.1 Ground ring $\mathbb{ZP}$

To define a cluster algebra $\mathcal{A}$ we must first fix its ground ring.

Let $(\mathbb{P}, \cdot)$ be a free abelian group (written multiplicatively) on variables $y_1, \ldots, y_n$ and define an addition $\oplus$ in $\mathbb{P}$ by

$$\prod_j y_j^{a_j} \oplus \prod_j y_j^{b_j} = \prod_j y_j^{\min(a_j, b_j)}. \tag{1.1}$$

For example $y_1^2 y_2^{-3} y_3 \oplus 1 = y_2^{-3}$. Then $(\mathbb{P}, \oplus, \cdot)$ is a semifield[1], and is called *tropical semifield*.

Let $\mathbb{ZP}$ denote the group ring of $\mathbb{P}$. Then $\mathbb{ZP}$ is the ring of Laurent polynomials in the variables $y_1, \ldots, y_n$. The ring $\mathbb{ZP}$ will be the ground ring for the cluster algebra.

[1]This means that $\oplus$ is commutative, associative and distributive with respect to the multiplication in $\mathbb{P}$.

Remark 1.1 *If this is the first time you see cluster algebras, then you may consider the special case where $\mathbb{P} = 1$, and $\mathbb{ZP} = \mathbb{Z}$ is just the ring of integers. In this case, we say the cluster algebra has* trivial coefficients.

Let $\mathbb{QP}$ denote the field of fractions of $\mathbb{ZP}$ and let $\mathcal{F} = \mathbb{QP}(x_1, \ldots, x_n)$ be the field of rational functions in n variables and coefficients in $\mathbb{QP}$.

Remark 1.2 *In the case of trivial coefficients, we have $\mathbb{QP} = \mathbb{Q}$.*

1.2 Seeds and mutations

The cluster algebra is determined by the choice of an initial seed $(\mathbf{x}, \mathbf{y}, Q)$, which consists of the following data.

- Q is a quiver without loops $\circ\!\circlearrowright$ and 2-cycles $\circ \rightleftarrows \circ$, and with n vertices;
- $\mathbf{y} = (y_1, \ldots, y_n)$ is the n-tuple of generators of $\mathbb{P}$, called *initial coefficient tuple*;
- $\mathbf{x} = (x_1, \ldots, x_n)$ is the n-tuple of variables of $\mathcal{F}$, called *initial cluster*.

The triple $(\mathbf{x}, \mathbf{y}, Q)$ is called the *initial seed* of the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$.

The cluster algebra is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by so-called *cluster variables*, and these cluster variables are constructed from the initial seed by a recursive method called *mutation*. A mutation transforms a seed $(\mathbf{x}, \mathbf{y}, Q)$ into a new seed $(\mathbf{x}', \mathbf{y}', Q')$. Given any seed there are n different mutations $\mu_1, \ldots, \mu_n$, one for each vertex of the quiver, or equivalently, one for each cluster variable in the cluster.

The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, \mathbf{y}, Q)$ into the seed $\mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}', \mathbf{y}', Q')$ defined as follows:

- $\mathbf{x}'$ is obtained from $\mathbf{x}$ by replacing one cluster variable by a new one, $\mathbf{x}' = \mathbf{x} \backslash \{x_k\} \cup \{x'_k\}$, and x'_k is defined by the following *exchange relation*

  $$x_k x'_k = \frac{1}{y_k \oplus 1} \left(y_k \prod_{i \to k} x_i \; + \; \prod_{i \leftarrow k} x_i \right) \tag{1.2}$$

  where the first product runs over all arrows in Q that end in k and the second product runs over all arrows that start in k.

- $\mathbf{y}' = (y'_1, \ldots, y'_n)$ is a new coefficient n-uple, where

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j \prod_{k \to j} y_k (y_k \oplus 1)^{-1} \prod_{k \leftarrow j} (y_k \oplus 1) & \text{if } j \neq k. \end{cases}$$

  Note that one of the two products is always trivial, hence equal to 1, since Q has no oriented 2-cycles. Also note that $\mathbf{y}'$ depends only on $\mathbf{y}$ and Q.

- The quiver Q' is obtained from Q in three steps:

  1. for every path $i \to k \to j$ add one arrow $i \to j$,
  2. reverse all arrows at k,
  3. delete 2-cycles.

See Figure 1.1 for three examples of quiver mutations.

Lemma 1.3 *Mutations are involutions, that is,* $\mu_k \mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}, \mathbf{y}, Q)$.

Note that Q' only depends on Q, that $\mathbf{y}'$ depends on $\mathbf{y}$ and Q, and that $\mathbf{x}'$ depends on the whole seed $(\mathbf{x}, \mathbf{y}, Q)$.

It is convenient to picture the mutation procedure in the so-called *exchange graph*. The vertices of this graph are the seeds of the cluster algebra and the edges are the mutations. Since we can always mutate in n directions, each vertex in the exchange graph has exactly n neighbors. See Figure 1.2 for an example with $n = 3$. The initial seed is one of the vertices in this graph. Applying the first n mutations to this seed, will producet the n neighbors of this vertex in the graph, each of which contains exactly one new cluster variable. So at this stage we have $2n$ cluster variables. Now we can continue mutating these new seeds, and at every step we construct a "new" cluster variable. It may happen, that we obtain a seed that has already appeared previously in this process. In that case we identify the two corresponding vertices in the n-regular graph, and the actual exchange graph is a quotient of the graph in Figure 1.2. Such a repetition may happen but it does not have to, and in general the number of seeds is infinite. The whole pattern is determined by the initial seed.

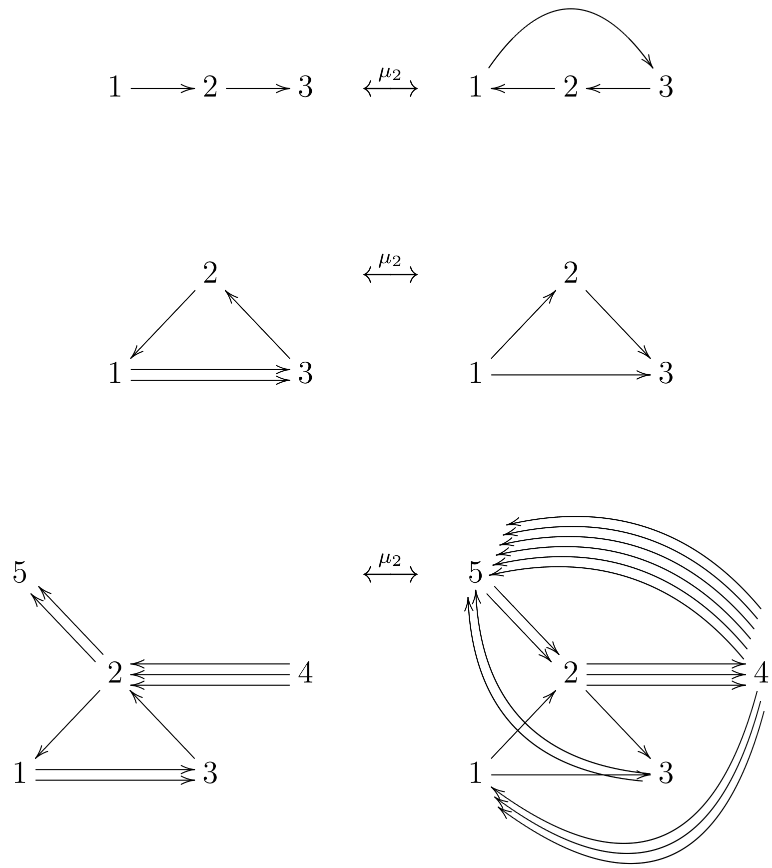


Figure 1.1: Examples of quiver mutations

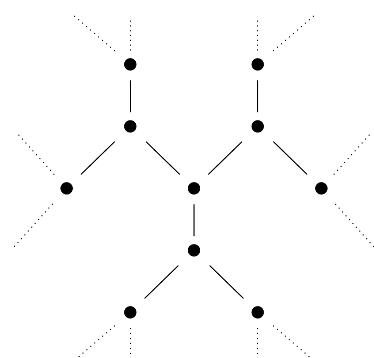


Figure 1.2: A 3-regular graph

1.3 Definition

Let $\mathcal{X}$ be the set of all cluster variables obtained by mutation from $(\mathbf{x}, \mathbf{y}, Q)$. The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$.

By definition, the elements of $\mathcal{A}$ are polynomials in $\mathcal{X}$ with coefficients in $\mathbb{ZP}$, so $\mathcal{A} \subset \mathbb{ZP}[\mathcal{X}]$. On the other hand, $\mathcal{A} \subset \mathcal{F}$, so the elements of $\mathcal{A}$ are also rational functions in $x_1, \ldots, x_n$ with coefficients in $\mathbb{QP}$.

Remark 1.4 Fomin and Zelevinsky define cluster algebras in a more general setting using skew-symmetrizable matrices instead of quivers. The quiver definition corresponds to the special case where the matrices are skew-symmetric.

1.4 Example $1 \to 2$

Let $(\mathbf{x}, \mathbf{y}, Q) = ((x_1, x_2), (1, 1), 1 \to 2)$.

Since the coefficients in this example are trivial, the coefficients in any seed will be $\{1, 1\}$. We therefore omit them in the computation below. Start with the initial seed.

$$(x_1, x_2), 1 \to 2$$

Apply mutation μ_1.

$$\left(\frac{x_2+1}{x_1}, x_2\right), 1 \leftarrow 2$$

Apply mutation μ_2.

$$\left(\frac{x_2+1}{x_1}, \frac{x_2+1+x_1}{x_1x_2}\right), 1 \to 2$$

Apply mutation μ_1. Let us do this step in detail. We get

$$\frac{\frac{x_2+1+x_1}{x_1x_2}+1}{\frac{x_2+1}{x_1}} = \frac{(x_2+1+x_1+x_1x_2)x_1}{x_1x_2(x_2+1)} = \frac{(x_2+1)(x_1+1)}{x_2(x_2+1)} = \frac{x_1+1}{x_2}.$$

Note that the denominator is a monomial. Thus the new seed is

$$\left(\frac{x_1+1}{x_2}, \frac{x_2+1+x_1}{x_1x_2}\right), 1 \leftarrow 2$$

Apply mutation μ_2.

$$\left(\frac{x_1+1}{x_2}, x_1\right), 1 \to 2$$

Apply mutation μ_1.

$$(x_2, x_1), 1 \leftarrow 2$$

Continuing the process from here will not yield new cluster variables. Thus in this case, there are exactly 5 cluster variables

$$x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_2+1+x_1}{x_1x_2}, \frac{x_1+1}{x_2}.$$

1.5 Example

Now consider the quiver $Q = 1 \rightrightarrows 2 \rightrightarrows 3$. Let us rather write it as $1 \xrightarrow{2} 2 \xrightarrow{2} 3$, where the number on the arrow from i to j indicates the number of arrows from i to j. Again we use trivial coefficients, so our initial seed is

$$(x_1, x_2, x_3),\ 1 \xrightarrow{2} 2 \xrightarrow{2} 3\,.$$

Apply mutation in 2.

$$\left(x_1, \frac{x_1^2+x_3^2}{x_2}, x_3\right),\ \begin{tikzcd} 1 \arrow[rr, bend left, "4"] & 2 \arrow[l, "2"'] & 3 \arrow[l, "2"'] \end{tikzcd}\,.$$

Apply mutation in 1.

$$\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}, \frac{x_1^2+x_3^2}{x_2}, x_3\right),\ \begin{tikzcd} 1 \arrow[r, "2"] & 2 \arrow[r, "6"] & 3 \arrow[ll, bend right, "4"'] \end{tikzcd}\,.$$

Apply mutation in 3.

$$\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}, \frac{x_1^2+x_3^2}{x_2}, \frac{\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}\right)^4+\left(\frac{x_1^2+x_3^2}{x_2}\right)^6}{x_3}\right),\ \begin{tikzcd} 1 \arrow[rr, bend left, "4"] & 2 \arrow[l, "22"'] & 3 \arrow[l, "6"'] \end{tikzcd}\,.$$

Apply mutation in 2. Then the new variable and the new quiver are

$$\frac{\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}\right)^{22}+\left(\frac{\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}\right)^4+\left(\frac{x_1^2+x_3^2}{x_2}\right)^6}{x_3}\right)^6}{\frac{x_1^2+x_3^2}{x_2}}, \quad \begin{tikzcd} 1 \arrow[r, "22"] & 2 \arrow[r, "6"] & 3 \arrow[ll, bend right, "128"'] \end{tikzcd}.$$

It is probably clear by now that each new cluster variable we obtain in this example is more complicated than the previous ones. Thus this cluster algebra has infinitely many cluster variables. It is also clear that the quivers we produce will have more and more arrows, and thus there are also infinitely many quivers in this example. Finally, you should be convinved by now that computations in cluster algebras are rather involved in general.

1.6 Laurent phenomenon and positivity

Theorem 1.5 [FZ1] *Let $u \in \mathcal{X}$ be any cluster variable. Then*

$$u = \frac{f(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $f \in \mathbb{ZP}[x_1, \dots, x_n], d_i \in \mathbb{Z}$.

Remark 1.6 This is a surprising result, since, a priori, the cluster variables are rational functions in the variables $x_1, \dots, x_n$. The theorem says that the denominators of these rational functions are actually monomials. This means that at each mutation, when we have to divide a binomial of cluster variables by a certain cluster variable x', the numerator of that cluster variable x' is actually a factor of that binomial. Note that the numerator of x' may be a complicated polynomial. We have already observed this phenomenon in the third step of Example 1.4. Try to see this phenomenon in the last step of Example 1.5.

Moreover we have the following positivity result.

Theorem 1.7 [LS] *The coefficients of the Laurent polynomials in Theorem 1.5 are positive in the sense that $f \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_1, \dots, x_n]$.*

Remark 1.8 *This result is not obvious; and actually 13 years have passed between the proof of Theorem 1.5 and the proof of Theorem 1.7. Although the binomial exchange relations only involve positive terms, one has to make sure that positivity is preserved when reducing the rational functions that one obtains in the mutation procedure to the Laurent polynomials in the theorems. This is not true for arbitrary rational functions as the example* $\frac{x^3+1}{x+1} = x^2 - x + 1$ *shows.*

1.7 Classifications

There are several special types of cluster algebras that have been studied using very different methods. We define these types here and show how they overlap.

We say that two quivers Q, Q' are mutation equivalent, and write $Q \sim Q'$, if there exists a finite sequence of muations transforming Q into Q'.

Definition 1.9 *A cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ *is said to be of*

(a) finite type *if the number of cluster variables is finite;*

(b) finite mutation type *if the number of quivers* Q' *that are mutation equivalent to* Q *is finite;*

(c) acyclic type *if* Q *is mutation equivalent to a quiver without oriented cycles;*

(d) surface type *if* Q *is the adjacency quiver of a triangulation of a marked surface (see Chapter 2).*

The cluster algebra of Example 1.4 is of finite type, finite mutation type, acyclic type and surface type. On the other hand, the cluster algebra of Example 1.5 is not of finite type, not of finite mutation type, not of surface type, but it is of acyclic type.

Fomin and Zelevinsky showed that the finite-type cluster algebras are classified by the Dynkin diagrams. Since we are considering only cluster algebras given by quivers, we only get the simply laced Dynkin diagrams.

Theorem 1.10 [FZ2] *The cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ *is of finite type if and only if* Q *is mutation equivalent to a quiver of Dynkin type* $\mathbb{A}, \mathbb{D}$ *or* $\mathbb{E}$*.*

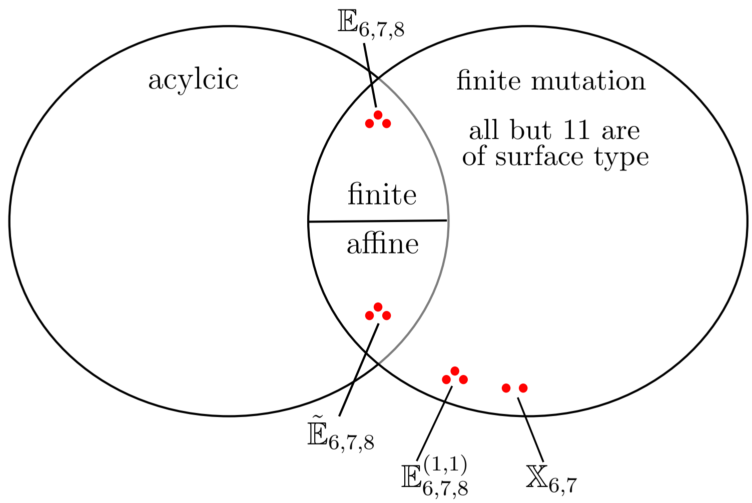


Figure 1.3: Different types of cluster algebras of rank $n \geq 3$

Finite mutation type is more general than finite type. The finite mutation type classification is due to Felikson, Shapiro and Tumarkin.

Theorem 1.11 [FeShTu] *The cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is of finite mutation type if and only if*

- *it is of surface type or*
- $n \leq 2$, *or*
- *it is one of 11 exceptional types* $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8, \mathbb{E}_6^{(1,1)}, \mathbb{E}_7^{(1,1)}, \mathbb{E}_8^{(1,1)}, \mathbb{X}_6, \mathbb{X}_7$.

The overlaps of the various classes of cluster algebras for $n \geq 3$ are illustrated in Figure 1.3. The accylic and the mutation finite types have in common the finite types $\mathbb{A}, \mathbb{D}, \mathbb{E}$ and the tame types corresponding to the extended Dynkin diagrams $\widetilde{\mathbb{A}}, \widetilde{\mathbb{D}}, \widetilde{\mathbb{E}}$. Other acyclic types are called wild. The 11 exceptions in Theorem 1.11 are indicated by dots; 9 of them correspond to root systems of certain $\mathbb{E}$-types. The other 2 types $\mathbb{X}_6, \mathbb{X}_7$ had not appeared elsewhere before this classification.

Chapter 2

Cluster algebras of surface type

Building on work of Fock and Goncharov [FG1, FG2], and of Gekhtman, Shapiro and Vainshtein [GSV], Fomin, Shapiro and Thurston [FST] associated a cluster algebra to any *bordered surface with marked points*.

2.1 Marked surfaces

We fix the following notation.

- S is a connected oriented Riemann surface with (possibly empty) boundary ∂S.
- $M \subset S$ is a finite set of *marked points* with at least one marked point on each connected component of the boundary.

We will refer to the pair (S, M) simply as a *surface*. A surface is called *closed* if the boundary is empty. Marked points in the interior of S are called *punctures*. Examples are shown in Figure 2.1. For technical reasons, we require that (S, M) is not a sphere with 1, 2 or 3 punctures; a monogon with 0 or 1 puncture; or a bigon or triangle without punctures.

Remark 2.1 *in Chapters 3 and 4, we will restrict to surfaces without punctures. The reason for this restriction in Chapter 3 is only for the sake of simplicity, but in Chapter 4 it is necessary. In the appendix, we explain how to modify the results in Chapter 3 in the presence of punctures.*

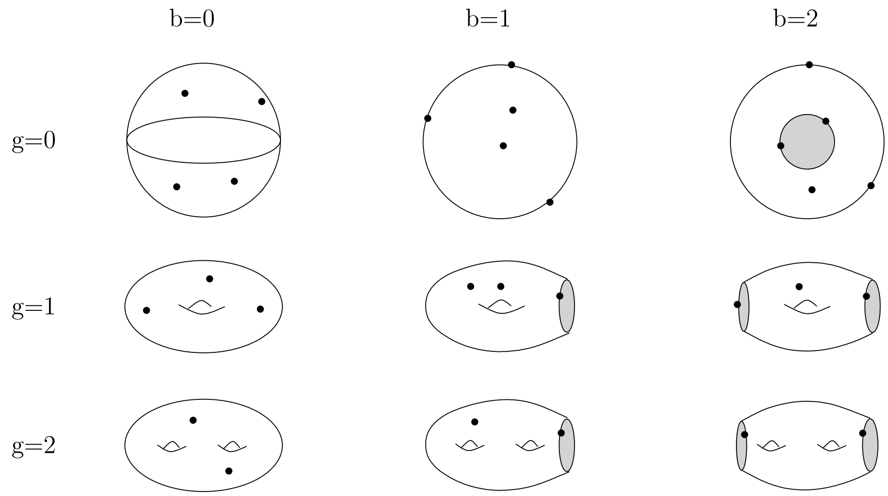


Figure 2.1: Examples of surfaces, g is the genus and b the number of boundary components

2.2 Arcs and triangulations

An *arc* γ in (S, M) is a curve in S, considered up to isotopy[1], such that

(a) the endpoints of γ are in M;

(b) except for the endpoints, γ is disjoint from M and from ∂S,

(c) γ does not cut out an unpunctured monogon or an unpunctured bigon;

(d) γ does not cross itself, except that its endpoints may coincide.

A *generalized arc* is a curve which satisfies conditions (a),(b) and (c), but it can have selfcrossings. Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are called *boundary segments*. By (c), boundary segments are not arcs. A *closed loop* is a closed curve in S which is disjoint from the boundary of S.

[1] A *homotopy* between two continuous maps $f, g\colon X \to Y$ is a continuous map $h\colon [0,1] \times X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$. An *isotopy* is a homotopy h such that for all $t \in [0,1]$ the map $h(t,-)\colon X \to h(t,X)$ is a homeomorphism. In particular an isotopy of curves cannot create selfcrossings.

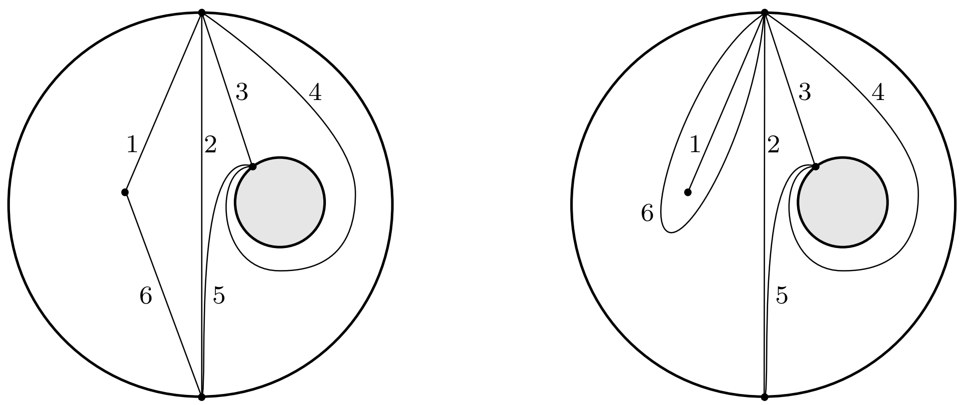


Figure 2.2: Two ideal triangulations of a punctured annulus related by a flip of the arc 6. The triangulation on the right hand side has a self-folded triangle.

For any two arcs γ, γ' in S, define

$$e(\gamma, \gamma') = \min\{\text{number of crossings of } \alpha \text{ and } \alpha' \mid \alpha \simeq \gamma, \alpha' \simeq \gamma'\},$$

where α and α' range over all arcs isotopic to γ and γ', respectively. We say that arcs γ and γ' are *compatible* if $e(\gamma, \gamma') = 0$.

An *ideal triangulation* is a maximal collection of pairwise compatible arcs (together with all boundary segments). The arcs of a triangulation cut the surface into *ideal triangles*. Triangles that have only two distinct sides are called *self-folded* triangles. Note that a self-folded triangle consists of a loop ℓ, together with an arc r to an enclosed puncture which we call a *radius*. Examples of ideal triangulations are given in Figure 2.2 as well as in Figures 3.5 and 3.6.

Lemma 2.2 *The number of arcs in an ideal triangulation is exactly*

$$n = 6g + 3b + 3p + c - 6,$$

where g is the genus of S, b is the number of boundary components, p is the number of punctures and $c = |M| - p$ is the number of marked points on the boundary of S. The number n is called the rank *of (S, M).*

For the sake of completness, we include a proof of this fact, since it is usually omitted in the cited research papers.

Proof. Recall that the Euler characteristic of a surface S is given by $\chi(S) = f - e + v$, where v is the number of vertices, e is the number of edges, and f is the number of faces in any triangulation of S. By induction on the genus, one can show that for a closed surface $\chi(S) = 2 - 2g$. Moreover, if the boundary of S has b connected components then

$$\chi(S) = 2 - 2g - b, \tag{2.1}$$

since removing a disk from S can be thought of reducing the number of faces by one. Now consider a set of marked points M and a triangulation T. Then the number of vertices in T is $|M| = c + p$. The number of edges in T is the number of arcs n plus the number of boundary segments c. Thus

$$e = c + n \qquad \text{and} \qquad v = c + p. \tag{2.2}$$

We use induction on p. If $p = 0$, then each triangle has 3 distinct sides. Each of the n arcs lies in precisely 2 triangles and each of the c boundary segments lies in precisely 1 triangle. Therefore

$$3f = 2n + c. \tag{2.3}$$

Using equations (2.1)–(2.3) we get

$$\frac{2n + c}{3} - c - n + c = 2 - 2g - b,$$

and the statement follows.

Now suppose that $p > 0$. Let T be a triangulation of (S, M), and let us add a puncture x. Then we need to add 3 arcs to complete the triangulation. Indeed, the new puncture x lies in some triangle Δ of the old triangulation T and connecting x with the three vertices of Δ completes the triangulation. Thus adding a puncture increases n by 3. $\square$

Ideal triangulations are connected to each other by sequences of *flips*. Each flip replaces a single arc γ in T by a unique new arc $\gamma' \neq \gamma$ such that

$$T' = (T \setminus \{\gamma\}) \cup \{\gamma'\}$$

is a triangulation. See Figure 2.3.

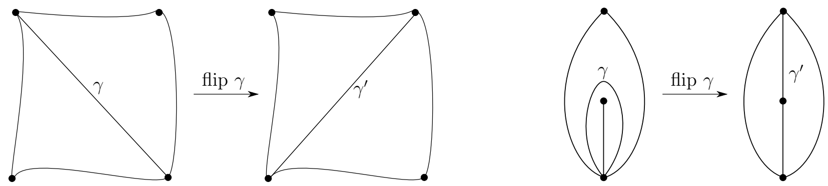


Figure 2.3: Two examples of flips

2.3 Cluster algebras from surfaces

We are now ready to define the cluster algebra associated to the surface. For that purpose, we choose an ideal triangulation $T = \{\tau_1, \tau_2, \ldots, \tau_n\}$ and then define a quiver Q_T without loops or 2-cycles as follows. The vertices of Q_T are in bijection with the arcs of T, and we denote the vertex of Q_T corresponding to the arc τ_i simply by i. The arrows of Q_T are defined as follows. For any triangle Δ in T which is not self-folded, we add an arrow $i \to j$ whenever

(a) τ_i and τ_j are sides of Δ with τ_j following τ_i in the clockwise order;

(b) τ_j is a radius in a self-folded triangle enclosed by a loop τ_ℓ, and τ_i and τ_ℓ are sides of Δ with τ_ℓ following τ_i in the clockwise order;

(c) τ_i is a radius in a self-folded triangle enclosed by a loop τ_ℓ, and τ_ℓ and τ_j are sides of Δ with τ_j following τ_ℓ in the clockwise order;

Then we remove all 2-cycles.

For example, the quiver corresponding to the triangulation on the right of Figure 2.2 is

$$\begin{tikzcd}
1 \arrow[dr] & & 3 \arrow[dr] \arrow[dd] & \\
& 2 \arrow[ur] & & 4 \\
6 \arrow[ur] & & 5 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

To define an initial seed, we associate an indeterminate x_i to each $\tau_i \in T$ and set the initial cluster $\mathbf{x}_T = (x_1, \ldots, x_n)$; and we set the initial coefficient tuple $\mathbf{y}_T = (y_1, \ldots, y_n)$ to be the vector of generators of $\mathbb{P}$. Then the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ is called the *cluster algebra associated to the surface* (S, M) *with principal coefficients in* T.

Fomin, Shapiro and Thurston showed that, up to a change of coefficients, the cluster algebra does not depend on the choice of the initial triangulation T. Moreover, they proved the following correspondence.

Theorem 2.3 [FST] *There are bijections*

$$\begin{array}{ccc} \{\text{cluster variables of } \mathcal{A}\} & \longleftrightarrow & \{\text{arcs of } (S,M)\} \\ x_\gamma & & \gamma \\[1em] \{\text{clusters of } \mathcal{A}\} & \longleftrightarrow & \{\text{triangulations of } (S,M)\} \\ \mathbf{x}_T = \{x_{\gamma_1}, \ldots, x_{\gamma_n}\} & & T = \{\gamma_1, \ldots, \gamma_n\} \end{array}$$

Moreover, if γ_k *is not the radius of a self-folded triangle in a triangulation* T*, then the mutation in* k *corresponds to the flip of the arc* γ_k*, that is, the cluster*

$$\mu_k(\mathbf{x}_T) = (\mathbf{x}_T \setminus \{x_{\gamma_k}\}) \cup \{x'_{\gamma_k}\}$$

corresponds to the triangulation

$$\mu_{\gamma_k}(T) = (T \setminus \{\gamma_k\}) \cup \{\gamma'_k\}.$$

Remark 2.4 *For simplicity, we excluded the case where* γ_k *is the radius of a self-folded triangle because then* γ_k *cannot be flipped. In [FST] the authors solve this problem by introducing tagged arcs and tagged triangulations, replacing the loop of a self-folded triangle by a second radius. In that setup the theorem holds without any restrictions.*

Remark 2.5 *If* β *is a boundary segment, we set* $x_\beta = 1$*.*

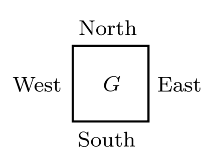
Chapter 3

Snake graphs and expansion formulas

Abstract snake graphs and band graphs were introduced and studied in [CS, CS2, CS3] motivated by the snake graphs and band graphs appearing in the combinatorial formulas for cluster algebra elements in [Pr, MS, MSW, MSW2]. Throughout we fix the standard orthonormal basis of the plane.

3.1 Snake graphs

A **tile** G is a square in the plane whose sides are parallel or orthogonal to the elements in the fixed basis. All tiles considered will have the same side length.



We consider a tile G as a graph with four vertices and four edges in the obvious way. A *snake graph* $\mathcal{G}$ is a connected planar graph consisting of a finite sequence of tiles $G_1, G_2, \ldots, G_d$, with $d \geq 1$, such that for each i, the tiles G_i and G_{i+1} share exactly one edge e_i, and this edge is either the north edge of G_i and the south edge of G_{i+1}, or it is the east edge of G_i and the west edge of G_{i+1}. An example of a snake graph with 8 tiles is given in Figure 3.1.

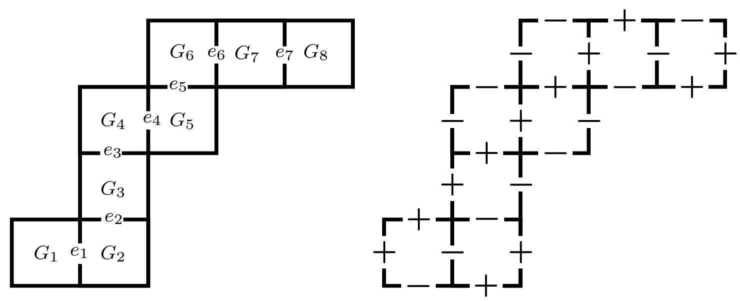


Figure 3.1: A snake graph with 8 tiles and 7 interior edges (left); a sign function on the same snake graph (right)

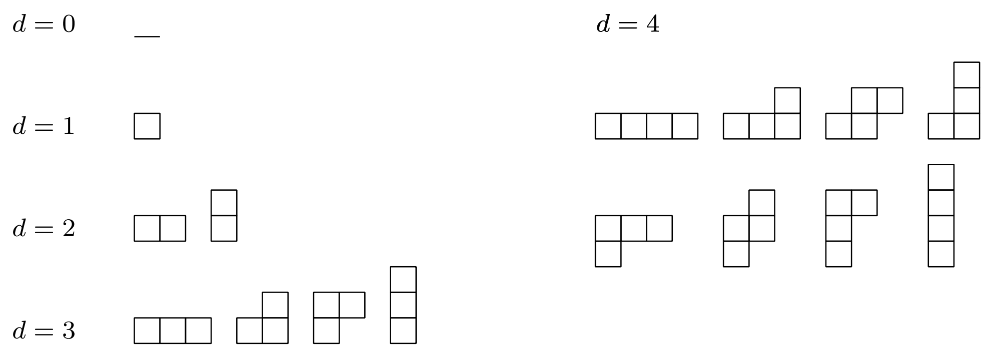


Figure 3.2: A list of all snake graphs with at most 4 tiles

The graph consisting of two vertices and one edge joining them is also considered a snake graph. Figure 3.2 lists all snake graphs with at most 4 tiles.

3.1.1 Some notation and terminology

Let $\mathcal{G} = (G_1, G_2, \ldots, G_d)$ be a snake graph.

- The $d-1$ edges $e_1, e_2, \ldots, e_{d-1}$ which are contained in two tiles are called *interior edges* of $\mathcal{G}$ and the other edges are called *boundary edges.* We will always use the natural ordering of the set of interior edges, so that e_i is the edge shared by the tiles G_i and G_{i+1}.

- Let $\operatorname{Int} \mathcal{G} = \{e_1, \ldots, e_{d-1}\}$ denote the set of all interior edges of $\mathcal{G}$.

- Let ${}_{SW}\mathcal{G}$, $\mathcal{G}^{NE}$ denote the following sets.

$$
\begin{aligned}
{}_{SW}\mathcal{G} &= \{ \text{ south edge of } G_1, \text{ west edge of } G_1 \ \}; \\
\mathcal{G}^{NE} &= \{ \text{ north edge of } G_d, \text{ east edge of } G_d \ \}.
\end{aligned}
$$

  If $\mathcal{G}$ is a single edge, we let ${}_{SW}\mathcal{G} = \emptyset$ and $\mathcal{G}^{NE} = \emptyset$.

- We say that two snake graphs are *isomorphic* if they are isomorphic as graphs.

3.1.2 Sign function

A *sign function* f on a snake graph $\mathcal{G}$ is a map

$$f\colon \{\text{edges of } \mathcal{G}\} \to \{+,-\},$$

such that on every tile in $\mathcal{G}$

- the north and the west edge have the same sign,
- the south and the east edge have the same sign
- the sign on the north edge is opposite to the sign on the south edge.

See Figure 3.1 for an example. Note that on every snake graph there are exactly two sign functions.

A snake graph is determined up to symmetry by its sequence of tiles together with a sign function on the interior edges.

3.2 Band graphs

Band graphs are obtained from snake graphs by identifying a boundary edge of the first tile with a boundary edge of the last tile, where both edges have the same sign. We use the notation $\mathcal{G}^\circ$ for general band graphs, indicating their circular shape, and we also use the notation $\mathcal{G}^b$ if we know that the band graph is constructed by glueing a snake graph $\mathcal{G}$ along an edge b.

More precisely, to define a band graph $\mathcal{G}^\circ$, we start with an abstract snake graph $\mathcal{G} = (G_1, G_2, \ldots, G_d)$ with $d \geq 1$, and fix a sign function on $\mathcal{G}$. Denote by x the southwest vertex of G_1, let $b \in {}_{SW}\mathcal{G}$ the south edge (respectively the west edge) of G_1, and let y denote the other endpoint of b, see Figure 3.3. Let b' be the unique edge in $\mathcal{G}^{NE}$ that has the same sign as b, and let y' be the northeast vertex of G_d and x' the other endpoint of b'.

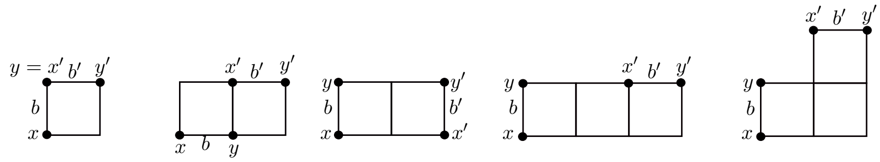
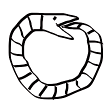


Figure 3.3: Examples of small band graphs; the two band graphs with 3 tiles are isomorphic.



Let $\mathcal{G}^b$ denote the graph obtained from $\mathcal{G}$ by identifying the edge b with the edge b' and the vertex x with x' and y with y'. The graph $\mathcal{G}^b$ is called a *band graph* or *ouroborus*[1]. Note that non-isomorphic snake graphs can give rise to isomorphic band graphs. See Figure 3.3 for examples.

The interior edges of the band graph $\mathcal{G}^b$ are by definition the interior edges of $\mathcal{G}$ plus the glueing edge $b = b'$. A band graph is uniquely determined by its sequence of tiles $G_1, \ldots, G_d$ together with its sign function on the interior edges (including the glueing edge).

In order to be able to formally take sums of band graphs we make the following definition.

Definition 3.1 *Let $\mathcal{R}$ denote the free abelian group generated by all isomorphism classes of finite disjoint unions of snake graphs and band graphs. If $\mathcal{G}$ is a snake graph, we also denote its class in $\mathcal{R}$ by $\mathcal{G}$, and we say that $\mathcal{G} \in \mathcal{R}$ is a* positive *snake graph and that its inverse $-\mathcal{G} \in \mathcal{R}$ is a* negative *snake graph.*

3.3 From snake graphs to surfaces

Given a snake graph $\mathcal{G} = (G_1, G_2, \ldots, G_d)$, we can construct a triangulated polygon as follows.

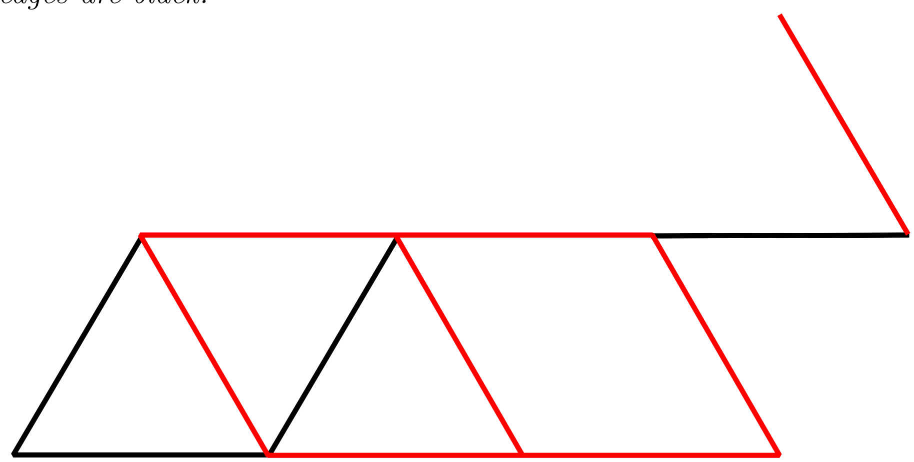
- In each tile G_i add a diagonal τ_i from the north west corner to the south east corner.
- Tilt the snake graph such that each tile becomes a parallelogram consisting of two equilateral triangles.

[1]Ouroboros: a snake devouring its tail.

- Fold the snake graph along the interior edges $e_1, \ldots, e_{d-1}$, and, at each folding, identify the two triangles on either side of the interior edge.

This produces a surface that is homeomorphic to a triangulated polygon with $d+3$ vertices whose set of boundary segments is precisely ${}_{SW}\mathcal{G} \cup \operatorname{Int}\mathcal{G} \cup \mathcal{G}^{NE}$ and whose triangulation is given by the diagonals $\tau_1, \ldots, \tau_d$.

Exercise 3.2 *Cut out the following figure and perform the folding. Interior edges are black.*



A *labeled* snake graph is a snake graph in which each edge and each tile carries a label or weight. For example, for snake graphs from cluster algebras of surface type, these labels are cluster variables.

3.4 Labeled snake graphs from surfaces

Now we want to go the other way and associate a snake graph to every arc in a triangulated surface.

Let T be an ideal triangulation of a surface (S, M) and let γ be an arc in (S, M) which is not in T. Choose an orientation on γ, let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. Denote by

$$s = p_0, p_1, p_2, \ldots, p_{d+1} = t$$

the points of intersection of γ and T in order. For $j = 1, 2, \ldots, d$, let τ_{i_j} be the arc of T containing p_j, and let Δ_{j-1} and Δ_j be the two ideal triangles in

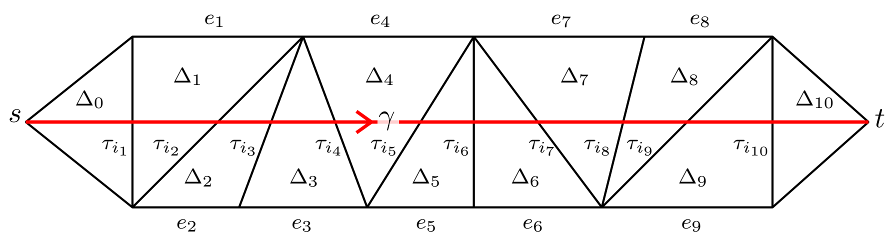


Figure 3.4: The arc γ passing through $d+1$ triangles $\Delta_0, \ldots, \Delta_d$

T on either side of τ_{i_j}. Then, for $j = 1, \ldots, d-1$, the arcs τ_{i_j} and $\tau_{i_{j+1}}$ form two sides of the triangle Δ_j in T and we define e_j to be the third arc in this triangle, see Figure 3.4.

Let G_j be the quadrilateral in T that contains τ_{i_j} as a diagonal. We will think of G_j as a *tile* as in section 3.1, but now the edges of the tile are arcs in T and thus are labeled edges. We also think of the tile G_j itself being labeled by the diagonal τ_{i_j}.

Define a *sign function* f on the edges $e_1, \ldots, e_d$ by

$$f(e_j) = \begin{cases} +1 & \text{if } e_j \text{ lies on the right of } \gamma \text{ when passing through } \Delta_j \\ -1 & \text{otherwise.} \end{cases}$$

The labeled snake graph $\mathcal{G}_\gamma = (G_1, \ldots, G_d)$ with tiles G_i and sign function f is called the *snake graph associated to the arc* γ. Each edge e of $\mathcal{G}_\gamma$ is labeled by an arc $\tau(e)$ of the triangulation T. We define the *weight* $x(e)$ of the edge e to be cluster variable associated to the arc $\tau(e)$. Thus $x(e) = x_{\tau(e)}$.

Note that we can define a sign function f in the same way for a any closed loop ζ. In that case we define the *band graph* $\mathcal{G}^\circ_\zeta$ of ζ to be the band graph with tiles G_i and sign function f.

3.5 Perfect matchings, height and weight

A *perfect matching* of a graph $\mathcal{G}$ is a subset P of the edges of $\mathcal{G}$ such that each vertex of $\mathcal{G}$ is incident to exactly one edge of P. We define

$$\operatorname{Match} \mathcal{G} = \{\text{perfect matchings of } \mathcal{G}\}.$$

If $\mathcal{G}^\circ = \mathcal{G}^b$ is a band graph, we define $\operatorname{Match} \mathcal{G}^\circ$ to be the set of all perfect matchings P of the snake graph $\mathcal{G}$ such that P is a perfect matching of $\mathcal{G}^\circ$.

Each snake graph $\mathcal{G}$ has precisely two perfect matchings P_-, P_+ that contain only boundary edges. We call P_- the minimal matching and P_+ the maximal matching of $\mathcal{G}$. [2]

$P_- \ominus P = (P_- \cup P) \setminus (P_- \cap P)$ denotes the symmetric difference of an arbitrary perfect matching $P \in \operatorname{Match}\mathcal{G}$ with the minimal matching P_-.

Definition 3.3 *Let $P \in \operatorname{Match}\mathcal{G}$. The set $P_- \ominus P$ is the set of boundary edges of a (possibly disconnected) subgraph $\mathcal{G}_P$ of $\mathcal{G}$, and $\mathcal{G}_P$ is a union of tiles*

$$\mathcal{G}_P = \bigcup_i G_i.$$

We define the height monomial *of P by*

$$y(P) = \prod_{G_i \text{ a tile in } \mathcal{G}_P} y_i$$

Thus $y(P)$ is the product of all y_i for which the tile G_i lies inside in $P \ominus P_-$ with multiplicities.

3.6 Expansion formula

Let $T = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of (S, M) and let $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ be the cluster algebra with principal coefficients at T. Thus $\mathbf{x}_T = (x_1, \ldots, x_n)$, $\mathbf{y}_T = (y_1, \ldots, y_n)$ and $\mathbb{P} = \operatorname{Trop}(y_1, \ldots, y_n)$.

For simplicity, we assume here that there are no self-folded triangles in T. For the general case see the appendix.

Theorem 3.4 [MSW] *Let γ be an arc not in the triangulation T. Then the cluster variable x_γ is equal to*

$$x_\gamma = \frac{1}{\operatorname{cross}(\gamma)} \sum_{P \in \operatorname{Match}\mathcal{G}_\gamma} x(P) y(P),$$

where $x(P) = \prod_{e \in P} x(e)$ is the weight of P, $y(P)$ is the height of P and cross$(\gamma) = \prod_{j=1}^{d} x_{i_j}$.

---

[2] There is a choice involved here which of the two is P_- and this will make a difference later when we consider expansion formulas for cluster algebras with non-trivial coefficients. One can determine P_- as follows. If a tile G_j has the same orientation as the surface S the P_- contains the south and the north edges of G_j if they are boundary edges, and P_- does not contain the east or the west edge of G_j.

Since this theorem gives us a direct formula for the cluster variables, it allows us to redefine the cluster algebra without using mutations as in the following corollary.

Corollary 3.5 *The cluster algebra $\mathcal{A}$ of the surface (S, M) with principal coefficients in the triangulation T is the $\mathbb{ZP}$ subalgebra of $\mathcal{F}$ generated by all*

$$\frac{1}{\text{cross}(\gamma)} \sum_{P \in \text{Match}\, \mathcal{G}_\gamma} x(P) y(P),$$

where γ runs over all arcs in (S, M).

3.7 Examples

In the example in Figure 3.5, we compute the snake graph $\mathcal{G}_\gamma$ of an arc γ in a triangulated polygon. The arc γ crosses two arcs of the triangulation, hence the snake graph $\mathcal{G}_\gamma$ has two tiles. The graph $\mathcal{G}_\gamma$ admits exactly 3 perfect matchings (drawn in red), and they form a linear poset in which P_- is the unique minimal element and P_+ is the unique maximal element. The corresponding monomials are listed in the rightmost column. Thus in this example we have

$$x_\gamma = \frac{x_1\, y_1 y_2 + x_3\, y_1 + x_2}{x_1 x_2}.$$

In the example in Figure 3.6, we compute the snake graph $\mathcal{G}_\gamma$ of an arc γ in a triangulated annulus. The arc γ crosses the triangulation three times, twice in the arc labeled 1 and once in the arc labeled 2. Hence the snake graph $\mathcal{G}_\gamma$ has two tiles labeled 1 and one tile labeled 2. The graph $\mathcal{G}_\gamma$ admits exactly 5 perfect matchings drawn in red in the poset. The corresponding monomials are listed on the right of the poset. Thus in this example we have

$$x_\gamma = \frac{x_1^2\, y_1^2 y_2 + y_1^2 + 2x_2^2\, y_1 + x_2^4}{x_1^2 x_2}.$$

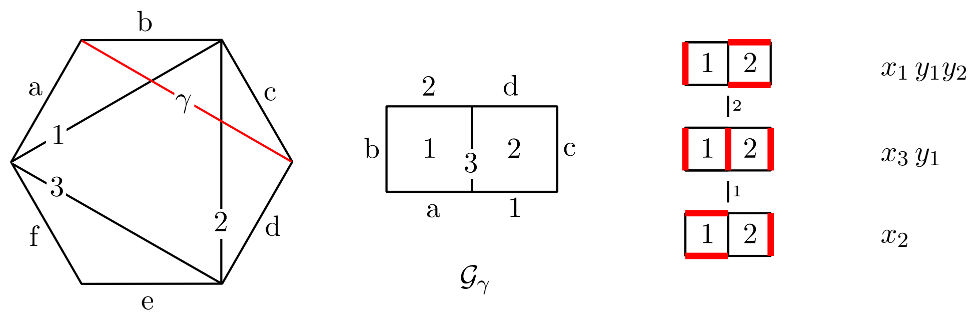


Figure 3.5: An arc γ in a triangulated hexagon (left), it's snake graph $\mathcal{G}_\gamma$ (center left) and its poset of perfect matchings (center right), and the corresponding monomials (right). The edges labeled a,b,c,d,e,f are boundary edges and their weights are one. The edges labeled 1,2,3 are arcs in the triangulation and their weights are the cluster variables x_1, x_2, x_3, respectively.

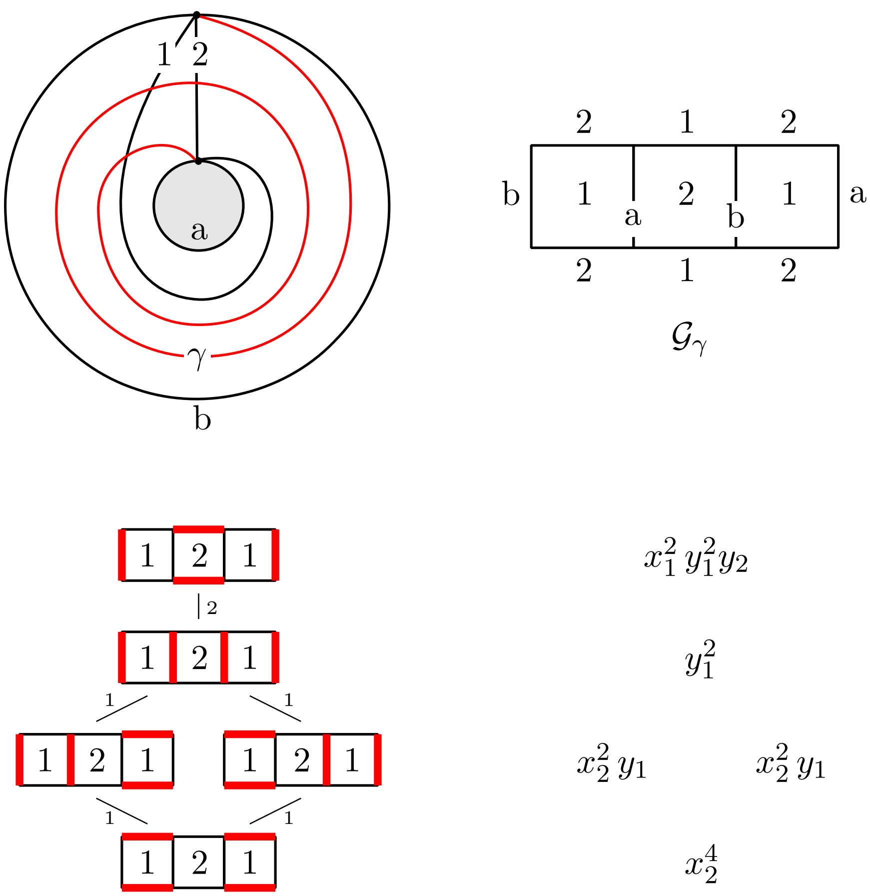


Figure 3.6: An arc γ in a triangulated annulus (top left), it's snake graph $\mathcal{G}_\gamma$ (top right), the poset of perfect matchings of $\mathcal{G}_\gamma$ (bottom left), and the corresponding monomials (bottom right). The edges labeled a,b are boundary edges and their weights are one. The edges labeled 1,2 are arcs in the triangulation and their weights are the cluster variables x_1, x_2, respectively.

Chapter 4

Bases for the cluster algebra

We would like to have a unique way to write each element of the cluster algebra as a sum of elements of a fixed basis. Recall the mutation exchange relations (1.2) among the cluster variables

$$(y_k \oplus 1)\, x_k x_k' = y_k \prod_{i \to k} x_i \ + \prod_{i \leftarrow k} x_i.$$

This is an example of an element of the cluster algebra that can be written in two different ways. We would like a *unique* way. In this section, we present two solutions to this problem for cluster algebras of surface type.

From now on let $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ be a cluster algebra arising from a surface (S, M) with principal coefficients at a fixed triangulation T, and assume that the surface has no punctures.[1] Since the cluster algebra is generated by cluster variables, we need to understand (sums of) products of cluster variables. Since cluster variables are in bijection with arcs, products of cluster variables are in bijection with sets of arcs with multiplicities. We call a set of curves with multiplicities a *multicurve*.

Moreover, to each arc we have associated a snake graph which allows us to compute the cluster variable via the perfect matching formula of Theorem 3.4. Therefore a product of cluster variables can be computed by the same perfect matching formula, replacing the single snake graph by a union of snake graphs. We get the following diagram

[1] So far, restricting to the case without punctures has been for the sake of simplicity. But now we really need to make this restriction.

$$\begin{tikzcd}
\text{product of cluster variables} \arrow[r] \arrow[d, equal] & \text{multicurve} \arrow[d] \\
\text{Laurent expansion} & \text{union of snake graphs} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\prod_{i=1}^{t} (x_{\gamma_i})^{\epsilon_i} \arrow[r] \arrow[d, equal] & \bigsqcup_{i=1}^{t} \bigsqcup_{j=1}^{\epsilon_i} \{\gamma_i\} \arrow[d] \\
\frac{1}{\mathrm{cross}(\mathcal{G})} \sum_{P \in \mathrm{Match}\,\mathcal{G}} x(P)y(P) & \mathcal{G} = \bigsqcup_{i=1}^{t} \bigsqcup_{j=1}^{\epsilon_i} \mathcal{G}_{\gamma_i} \arrow[l]
\end{tikzcd}$$

4.1 Skein relations

The relations among the cluster variables can be expressed on the level of arcs using smoothing operations [MW] and on the level of snake graphs as resolutions [CS, CS2, CS3].

Let γ_1 and γ_2 be two curves that cross at a point x. Then we define the *smoothing* of $\{\gamma_1, \gamma_2\}$ at x to be the pair of multicurves $\{\gamma_3, \gamma_4\}$ and $\{\gamma_5, \gamma_6\}$ obtained from $\{\gamma_1, \gamma_2\}$ by replacing the crossing $\times$ in a small neighborhood of x with the pair of segments $\asymp$ (respectively $\supset\subset$).

$$\{\gamma_1, \gamma_2\} \xrightarrow{\text{smoothing at } x} \{\gamma_3, \gamma_4\}\ ,\ \{\gamma_5, \gamma_6\}$$

If γ is a curve with a selfcrossing at a point x, we also define the *smoothing* of γ at x to be the pair of curves γ_{34} and γ_{56} obtained from γ by the same local transformation. See Figure 4.1 for examples of the smoothing operation.

It is important to notice that performing the smoothing operation on arcs may produced generalized arcs; and performing it on generalized arcs may produce closed loops. For example, the first smoothing step in Figure 4.1 produces a generalized arc, and the second step produces a closed loop.

A closed loop is called *essential* if it is not contractible and it has no selfcrossing.

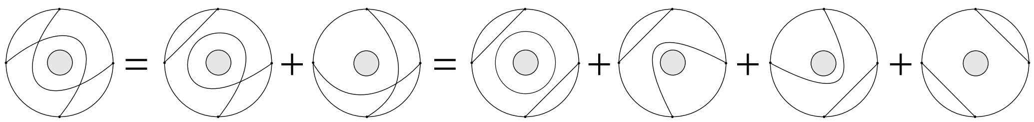


Figure 4.1: An example of the smoothing operation. The multicurve on the left has two crossings. Smoothing one of the crossings yields the sum of the two multicurves in the center. Both of them still have one crossing. Smoothing these yields the sum of the 4 multicurves on the right.

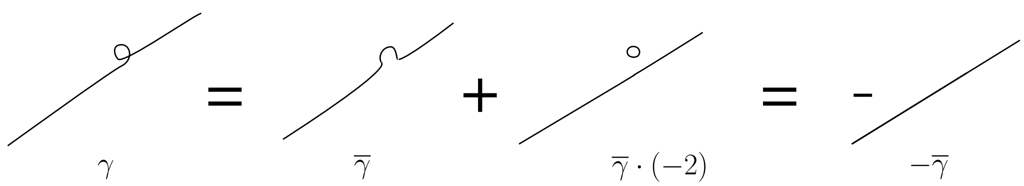


Figure 4.2: A curve γ with a contractible kink is equal to the negative of the same curve with the kink removed $\overline{\gamma}$.

If ζ is an essential loop, we use the perfect matching formula for the band graph $\mathcal{G}^\circ_\zeta$ of ζ to define a Laurent polynomial x_ζ. That is

$$x_\zeta = \frac{1}{\text{cross}(\zeta)} \sum_{P \in \text{Match}\, \mathcal{G}^\circ_\zeta} x(P)y(P).$$

If ζ is a contractible closed loop, we define $x_\zeta = -2$ (the integer -2). If γ is a curve that has a contractible kink then we set $x_\gamma = -x_{\overline{\gamma}}$, where $\overline{\gamma}$ is obtained from γ by contracting the kink to a point. Figure 4.2 illustrates that this definition is compatible with the smoothing of the selfcrossing at the kink.

Theorem 4.1 [MW]

(a) *If* $\{\gamma_3, \gamma_4\} \cup \{\gamma_5, \gamma_6\}$ *is obtained from* $\{\gamma_1, \gamma_2\}$ *by smoothing a crossing then*

$$x_{\gamma_1} x_{\gamma_2} = y_{34} x_{\gamma_3} x_{\gamma_4} + y_{56} x_{\gamma_5} x_{\gamma_6},$$

for some coefficients $y_{34}, y_{56} \in \mathbb{P}$.

(b) *If γ_{34}, γ_{56} is obtained from γ by smoothing a selfcrossing then*

$$x_\gamma = y_{34} x_{\gamma_{34}} + y_{56} x_{\gamma_{56}},$$

for some coefficients $y_{34}, y_{56} \in \mathbb{P}$.

Remark 4.2 (a) *The equations in Theorem 4.1 are called skein relations.*

(b) *Theorem 4.1 was proved in [MW] using hyperbolic geometry. A combinatorial proof in terms of snake and band graphs was given in [CS, CS2, CS3].*

4.1.1 Smoothing of arcs vs resolutions of snake graphs

The definition of smoothing is very simple. It is defined as a local transformation replacing a crossing $\times$ with the pair of segments $\asymp$ (resp. $\supset\subset$). But once this local transformation is done, one needs to find representatives inside the isotopy classes of the resulting curves which realize the minimal number of crossings with the fixed triangulation. This means that one needs to deform the obtained curves isotopically, and to 'unwind' them if possible, in order to see their actual crossing pattern, which is crucial for the applications to cluster algebras. This can be quite complicated especially in a higher genus surface.

The situation for the snake and band graphs is exactly opposite. The definition of the resolution is very complicated because one has to consider many different cases. But once all these cases are worked out, one has a complete list of rules in hand, which one can apply very efficiently in actual computations. The reason for this is that the definitions of the resolutions already take into account the isotopy mentioned above.

For explicit computations in the cluster algebra, one always needs to construct the snake graphs in order to compute the Laurent polynomials. Thus for this purpose it is more efficient to work with resolutions of snake graphs.

4.2 Definition of the bases $\mathcal{B}^\circ$ and $\mathcal{B}$.

We have seen that in order to define a basis, we need to understand products of cluster variables, and that a product of cluster variables corresponds to a

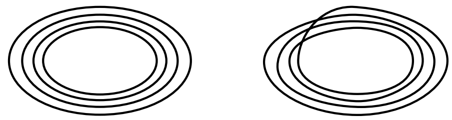


Figure 4.3: A bangle $\mathrm{Bang}_4\,\zeta$ on the left and a bracelet $\mathrm{Brac}_4\,\zeta$ on the right.

multicurve in the surface. Let k be the number of crossings in this multicurve. Then we can perform a smoothing operation at one of the crossings and obtain two multicurves, each of which will have at most $k-1$ crossings. Continuing this way, we can construct a collection of at most 2^k multicurves without crossings.

Now using the skein relations, we can perform each smoothing operation on the level of the Laurent polynomials. Thus we can express our original product of cluster variables as a sum of at most 2^k products $\prod_{\gamma\in C} x_\gamma$ where C is a multicurve without crossings.

We have also seen that the multicurves C that appear may contain arcs, boundary segments and closed loops. Below, we shall characterize the set $\mathcal{B}^\circ$ of all multicurves C that arise in this way. Then the above argument shows that $\mathcal{B}^\circ$ spans the cluster algebra.

We introduce the following notation. See Figure 4.3 for an illustration.

Definition 4.3 *Let ζ be an essential loop.*

(a) *The union of k copies of ζ is called the* k-bangle of ζ *and is denoted by* $\mathrm{Bang}_k\,\zeta$.

(b) *The closed loop obtained by concatenating ζ with itself k times is called the* k-bracelet of ζ *and is denoted by* $\mathrm{Brac}_k\,\zeta$.

Note that the k-bracelet $\mathrm{Brac}_k\,\zeta$ has exactly $k-1$ selfcrossings.

Definition 4.4 (a) *Let $\mathcal{B}^\circ$ be the set of all products $\prod_{\gamma\in C} x_\gamma$ where C ranges over all multicurves of arcs and essential closed loops without crossings.*

(b) *Let $\mathcal{B}$ be the set of all products $\prod_{\gamma\in C} x_\gamma$ where C ranges over all collections of arcs and bracelets such that*

- *no two elements of C cross, except for the selfcrossings of bracelets;*

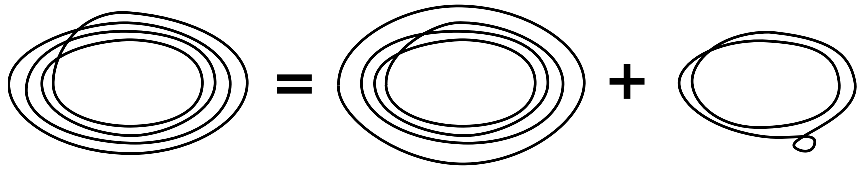


Figure 4.4: Skein relation showing that $\mathrm{Brac}_4(\zeta) = \zeta\,\mathrm{Brac}_3(\zeta) - \mathrm{Brac}_2(\zeta)$.

- *for every essential loop ζ, if $\mathrm{Brac}_k\,\zeta \in C$ then there is only one copy of it in C and no other bracelet of ζ is in C.*

Theorem 4.5 [MSW2] *Both $\mathcal{B}$ and $\mathcal{B}^\circ$ are bases for $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$.*

Proof idea: The fact that $\mathcal{B}^\circ$ spans the cluster algebra follows from the skein relations using the method described above. To show that $\mathcal{B}^\circ$ is linearly independent one uses the so called g-vectors of the cluster algebra elements, which is closely related to the sign functions of the snake graphs. Finally, one needs to show that the Laurent polynomials in $\mathcal{B}^\circ$ actually are elements of the cluster algebra. This is a surprisingly subtle point. In [MSW2] this was proved for unpunctured surfaces which have at least 2 marked points using the smoothing operations on arcs. In [CLS] the proof was extended to all unpunctured surfaces using snake graph calculus. This shows that $\mathcal{B}^\circ$ is a basis.

To prove that $\mathcal{B}$ is a basis, one needs to replace the bangles by the bracelets. Algebraically this can be done in terms of Chebyschev polynomials, $x_{\mathrm{Brac}_k\,\zeta} = T_k(x_\zeta)$ where T_k is the k-th Chebyshev polynomial of the second kind. These polynomials are defined recursively as

$$T_0(x) = 2,\ T_1(x) = x,\ T_k(x) = xT_{k-1}(x) - T_{k-2}(x),\ \text{for } k \geq 2.$$

Thus for the bracelets, we obtain the relation

$$\mathrm{Brac}_k(\zeta) = \zeta\,\mathrm{Brac}_{k-1}(\zeta) - \mathrm{Brac}_{k-2}(\zeta),$$

which can be seen also directly from the skein relations. In Figure 4.4, we illustrate the case $k = 4$, where we are smoothing the top crossing of the 4-bracelet. Note that the rightmost curve in that figure has a contractible kink, which produces the minus sign in the equation.

The basis $\mathcal{B}$ has the following important advantage over the basis $\mathcal{B}^\circ$.

Theorem 4.6 [T] *The basis $\mathcal{B}$ has positive structure constants.*

This means that if $b, b' \in \mathcal{B}$ are two basis elements, and if we express their product as a linear combination of elements in $\mathcal{B}$ as

$$bb' = \sum_{b'' \in \mathcal{B}} g_{b,b'}^{b''} b\text{''},$$

then the $g_{b,b'}^{b''} \in \mathbb{Z}\mathbb{P}$ are called the structure constants of the basis $\mathcal{B}$, and the theorem says that $g_{b,b'}^{b''} \in \mathbb{Z}_{\geq 0}\mathbb{P}$.

Remark 4.7 *The correspondence between the cluster algebra and a triangulated surface can be generalized to triangulated orbifolds, see [FeShTu]. In this setting the cluster algebra does not correspond to a quiver but to a skew-symmetrizable matrix, and, in contrast to the surface, the orbifold is allowed to have singularities. The results in chapters 3 and 4 have been genereralized to this setting in [FeShTu2, FeTu].*

Chapter 5

Appendix: Generalization to surfaces with punctures

5.1 Tagged arcs

Note that an arc γ that lies inside a self-folded triangle in T cannot be flipped. In order to rectify this problem, the authors of [FST] were led to introduce the slightly more general notion of *tagged arcs*.

A *tagged arc* is obtained by taking an arc that does not cut out a once-punctured monogon and marking ("tagging") each of its ends in one of two ways, *plain* or *notched*, so that the following conditions are satisfied:

- an endpoint lying on the boundary of S must be tagged plain
- both ends of a loop must be tagged in the same way.

Thus there are four ways to tag an arc between two distinct punctures and there are two ways to tag a loop at a puncture, see Figure 5.1. The notching is indicated by a bow tie.

One can represent an ordinary arc β by a tagged arc $\iota(\beta)$ as follows. If β does not cut out a once-punctured monogon, then $\iota(\beta)$ is simply β with both ends tagged plain. Otherwise, β is a loop based at some marked point q and cutting out a punctured monogon with the sole puncture p inside it. Let α be the unique arc connecting p and q and compatible with β. Then $\iota(\beta)$ is obtained by tagging α plain at q and notched at p.

Tagged arcs α and β are called *compatible* if and only if the following properties hold:

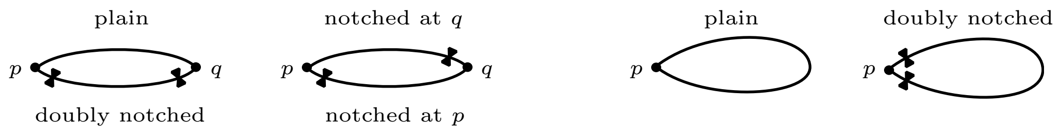


Figure 5.1: Four ways to tag an arc between two punctures (left); two ways to tag a loop at a puncture (right)

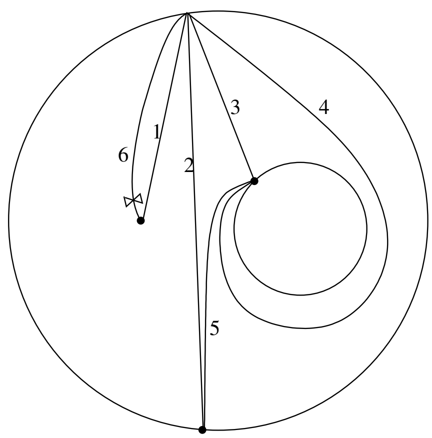


Figure 5.2: Tagged triangulation of the punctured annulus corresponding to the ideal triangulation of the right hand side of Figure 2.2.

- the arcs α^0 and β^0 obtained from α and β by forgetting the taggings are compatible;
- if $\alpha^0 = \beta^0$ then at least one end of α must be tagged in the same way as the corresponding end of β;
- $\alpha^0 \neq \beta^0$ but they share an endpoint a, then the ends of α and β connecting to a must be tagged in the same way.

A maximal collection of pairwise compatible tagged arcs is called a *tagged triangulation*. Figure 5.2 shows the tagged triangulation corresponding to the triangulation on the right hand side of Figure 2.2.

Given a surface (S, M) with a puncture p and a tagged arc γ, we let $\gamma^{(p)}$ denote the arc obtained from γ by changing its notching at p. If p and q are

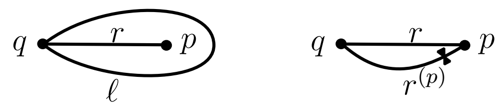


Figure 5.3: A self-folded triangle with loop ℓ and radius r (left); the corresponding tagged arcs r and $r^{(p)}$ (right). In the cluster algebra we have $x_\ell = x_r x_{r^{(p)}}$.

two punctures, we let $\gamma^{(pq)}$ denote the arc obtained from γ by changing its notching at both p and q.

If ℓ is an unnotched loop with endpoints at q cutting out a once-punctured monogon containing puncture p and radius r, see Figure 5.3 then we set

$$x_\ell = x_r x_{r^{(p)}}.$$

Thus the loop is equal to the product of the two radii.

5.2 Expansion formula for plain arcs in the presence of self-folded triangles

If there are self-folded triangles in the triangulation T then we have to modify the y-monomials in the expansion formula of Theorem 3.4 as follows. Recall that we had defined $y(P)$ as a monomial in the coefficients $y_1, \ldots, y_n$ and each y_i corresponds to an (untagged) arc y_{τ_i} of T. Now we need to redefine $y(P)$ by replacing every y_i in our previous definition by $\Phi(y_i)$, where Φ is defined below.

$$\Phi(y_i) = \begin{cases} y_i & \text{if } \tau_i \text{ is not a side of a self-folded triangle;} \\[2ex] \dfrac{y_r}{y_{r^{(p)}}} & \text{if } \tau_i \text{ is a radius } r \text{ to puncture } p \text{ in a self-folded triangle;} \\[2ex] y_{r^{(p)}} & \text{if } \tau_i \text{ is a loop } \ell \text{ in a self-folded triangle with radius } r \\ & \text{and puncture } p. \end{cases}$$

Then the cluster variable x_γ is equal to

$$x_\gamma = \frac{1}{\operatorname{cross}(\gamma)} \sum_{P \in \operatorname{Match} \mathcal{G}_\gamma} x(P) y(P).$$

5.3 Expansion formula for singly-notched arcs

Definition 5.1 *If p is a puncture, and $\gamma^{(p)}$ is a tagged arc with a notch at p but tagged plain at its other end, we define the associated crossing monomial as*

$$\operatorname{cross}(\gamma^{(p)}) = \frac{\operatorname{cross}(\ell_p)}{\operatorname{cross}(\gamma)} = \operatorname{cross}(\gamma) \prod_\tau x_\tau,$$

where the product is over all ends of arcs τ of T that are incident to p. If p and q are punctures and $\gamma^{(pq)}$ is a tagged arc with a notch at p and q, we define the associated crossing monomial as

$$\operatorname{cross}(\gamma^{(pq)}) = \frac{\operatorname{cross}(\ell_p)\operatorname{cross}(\ell_q)}{\operatorname{cross}(\gamma)^3} = \operatorname{cross}(\gamma) \prod_\tau x_\tau,$$

where the product is over all ends of arcs τ that are incident to p or q.

Let p be a puncture and let γ be an arc from a point $q \neq p$ to p. Let $\gamma^{(p)}$ be the tagged arc that is notched at p and plain at q and let ℓ denote the loop at q that cuts out the once-punctured monogon with puncture p and radius γ. Thus $\iota(\ell) = \gamma^{(p)}$. Let $\mathcal{G}_\ell$ be the snake graph of ℓ.

Definition 5.2 *The snake graph $\mathcal{G}_\ell$ contains two disjoint connected subgraphs, one on each end, both of which are isomorphic to $\mathcal{G}_\gamma$. We let $\mathcal{G}_{\gamma_p,1}$ denote the one at the south west end of $\mathcal{G}_\ell$ and $\mathcal{G}_{\gamma_p,2}$ the one at the north east end.*

We let $\mathcal{H}_{\gamma_p,1}$ be the subgraph of $\mathcal{G}_{\gamma_p,1}$ obtained by deleting the north east vertex, and $\mathcal{H}_{\gamma_p,2}$ be the subgraph of $\mathcal{G}_{\gamma_p,2}$ obtained by deleting the south west vertex.

In Figure 5.4, the subgraph $\mathcal{G}_{\gamma_p,1}$ is the subgraph of $\mathcal{G}_\ell$ consisting of the first two tiles and $\mathcal{G}_{\gamma_p,2}$ is the subgraph consisting of the last two tiles. In the same figure, the graph $\mathcal{H}_{\gamma_p,1}$ is the graph consisting of the first tile and the south edge of the second tile.

Definition 5.3 *A perfect matching P of $\mathcal{G}_\ell$ is called* γ-symmetric *if the restrictions of P to the two ends satisfy $P|_{\mathcal{H}_{\gamma_p,1}} \cong P|_{\mathcal{H}_{\gamma_p,2}}$.*

If P is γ-symmetric, define

$$\overline{x}(P) = \frac{x(P)}{x(P|_{\mathcal{G}_{\gamma,i}})}, \quad \overline{y}(P) = \frac{y(P)}{y(P|_{\mathcal{G}_{\gamma,i}})},$$

where $i = 1$ or 2 depending on which subgraph the restriction of P defines a perfect matching.

Let $T = \{\tau_1, \ldots, \tau_n\}$ be a tagged triangulation of (S, M) and let $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ be the cluster algebra with principal coefficients at T. Thus $\mathbf{x}_T = (x_1, \ldots, x_n)$, $\mathbf{y}_T = (y_1, \ldots, y_n)$ and $\mathbb{P} = \mathrm{Trop}(y_1, \ldots, y_n)$.

Let p be a puncture and assume that T contains no arc notched at p. In fact this is not really a restriction, because if the arcs in T are notched at p we can change the tags of all arcs at p to 'plain' and obtain the same quiver Q_T.

Let γ be an arc from a point $q \neq p$ to p and let $\gamma^{(p)}$ and ℓ be as above.

Theorem 5.4 *If γ is not in the triangulation T, then the cluster variable $x_{\gamma^{(p)}}$ is equal to*

$$x_{\gamma^{(p)}} = \frac{1}{\mathrm{cross}(\gamma^{(p)})} \sum_P \overline{x}(P)\, \overline{y}(P),$$

where the sum is over all γ-symmetric matchings P of $\mathcal{G}_\ell$.

Remark 5.5 *If γ is in T (so x_γ is an initial cluster variable), then since $x_{\gamma^{(p)}} = x_\ell / x_\gamma$, where x_ℓ is computed by the formula in Theorem 3.4.*

5.4 Expansion formula for doubly-notched arcs

For the case of a tagged arc with notches at both ends, we need two more definitions.

Let p and q be a punctures and let γ be an arc from a point p to q. Let $\gamma^{(p)}$ be the tagged arc that is notched at p and plain at q and let $\gamma^{(q)}$ be the tagged arc that is notched at q and plain at p. Let ℓ_p be the loop at q such that $\iota(\ell_p) = \gamma^{(p)}$, and let ℓ_q be the loop at p such that $\iota(\ell_q) = \gamma^{(q)}$.

Definition 5.6 *Assume that the tagged triangulation T does not contain either γ, $\gamma^{(p)}$, or $\gamma^{(q)}$. Let P_p and P_q be γ-symmetric matchings of $\mathcal{G}_{\ell_p}$ and $\mathcal{G}_{\ell_q}$, respectively. Then the pair (P_p, P_q) is called* γ-compatible *if at least one of the following two conditinos holds.*

- *The restrictions $P_p|_{\mathcal{G}_{\gamma_p,1}}$, and $P_q|_{\mathcal{G}_{\gamma_q,1}}$. are isomorphic perfect matchings of the subgraph $\mathcal{G}_{\gamma_p,1} \cong \mathcal{G}_{\gamma_q,1}$, or*

- *the restrictions* $P_p|_{\mathcal{G}_{\gamma_p,2}}$*, and* $P_q|_{\mathcal{G}_{\gamma_q,2}}$*. are isomorphic perfect matchings of the subgraph* $\mathcal{G}_{\gamma_p,2} \cong \mathcal{G}_{\gamma_q,2}$

If (P_p, P_q) *is a* γ*-compatible pair of matchings define the weight and height monomial,*

$$\overline{\overline{x}}(P_p, P_q) = \frac{x(P_p)\, x(P_q)}{x(P_p|_{\mathcal{G}_{\gamma_p,i}})^3}, \quad \overline{\overline{y}}(P_p, P_q) = \frac{y(P_p)\, y(P_q)}{y(P_p|_{\mathcal{G}_{\gamma_p,i}})^3},$$

where $i = 1$ *or* 2 *depending on the two cases above.*

For technical reasons, we require the (S, M) is not a closed surface with exactly 2 marked points for Theorem 5.7.

Theorem 5.7 *If* γ *is not in the triangulation* T*, then the cluster variable* $x_{\gamma^{(p)}}$ *is equal to*

$$x_{\gamma^{(pq)}} = \frac{1}{\operatorname{cross}(\gamma^{(pq)})} \sum_{(P_p,P_q)} \overline{\overline{x}}(P_p, P_q)\, \overline{\overline{y}}(P_p, P_q),$$

where the sum is over all γ*-compatible pairs of matchings* (P_p, P_q) *of* $(\mathcal{G}_{\ell_p}, \mathcal{G}_{\ell_q})$*.*

5.5 Example of a cluster expansion for a singly-notched arc

To compute the Laurent expansion of $x_{\gamma^{(p)}}$ of the notched arc in top left picture in Figure 5.4, we draw the snake graph $\mathcal{G}_\ell$ of the loop ℓ, shown in the top right picture of the same figure. The poset of γ-symmetric matchings of $\mathcal{G}_\ell$ is shown in the bottom left picture. Note that the matchings agree on the subgraphs $\mathcal{H}_{\gamma_p,1}$ and $\mathcal{H}_{\gamma_p,2}$. The corresponding monomials $\overline{x}(P)\overline{y}(P)$ are shown in the bottom right of the figure.

Simplifying and dividing by $\operatorname{cross}(\gamma^{(p)}) = x_1x_2x_3x_4$ we obtain

$$x_{\gamma^{(p)}} = \frac{x_1x_2x_3\, y_1y_2y_3y_4 + x_1x_3\, y_1y_2y_3 + x_4\, y_1y_3 + x_2x_4\, y_3 + x_2x_4\, y_1 + x_2^2x_4}{x_1x_2x_3x_4}.$$

Since all the initial variables and coefficients appearing in this sum correspond to ordinary arcs, no specialization of x-weights or y-weights was necessary in this case.

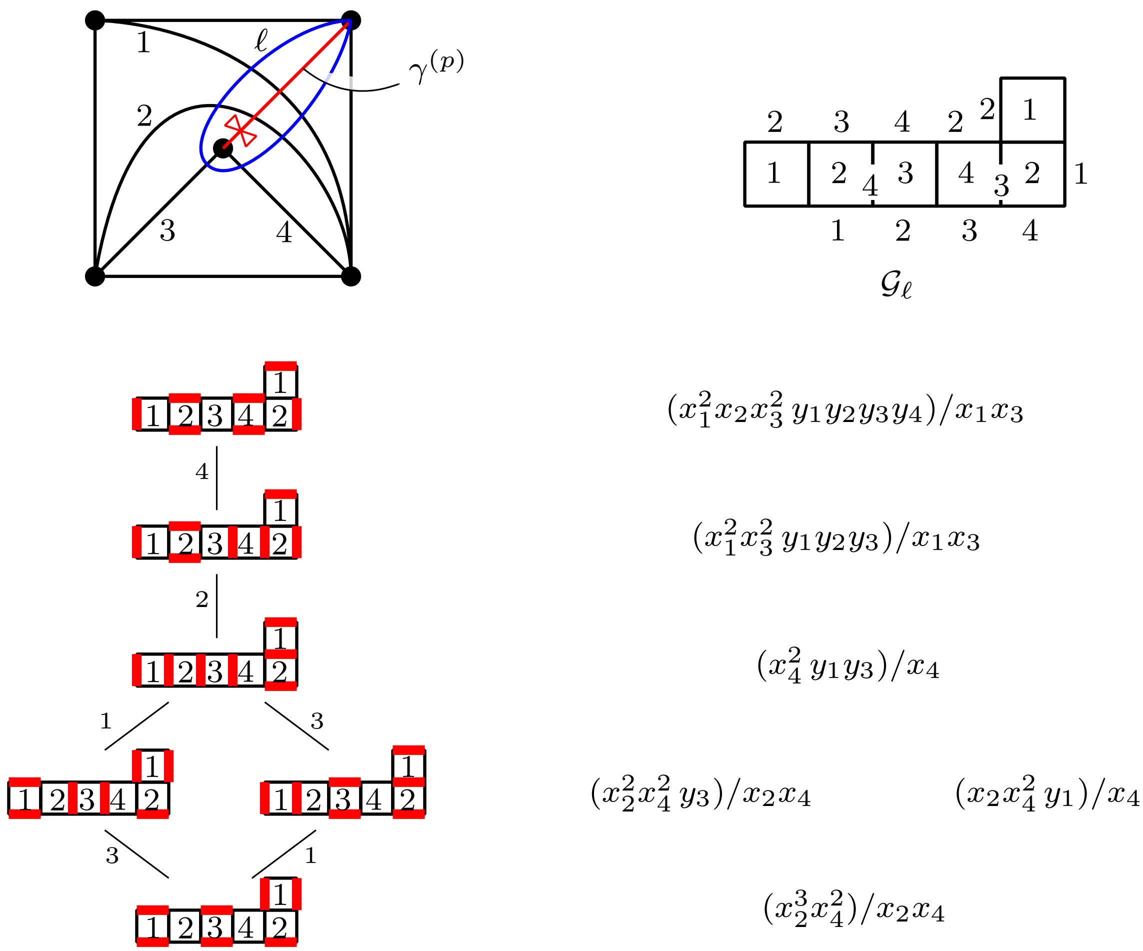


Figure 5.4: A notched arc $\gamma^{(p)}$ in a triangulated punctured square (top left), the snake graph $\mathcal{G}_\ell$ of the corresponding loop ℓ (top right) the poset of γ-symmetric matchings of $\mathcal{G}_\ell$ (bottom left) and the corresponding monomials (bottom right).

5.6 Example of a Laurent expansion for a doubly-notched arc

We close with an example of a cluster expansion formula for a tagged arc with notches at both endpoints. The top left picture in Figure 5.5. shows a triangulation of a sphere with 6 punctures in the shape of an octahedron. The numbers 1,...,12 in that figure are the labels of the arcs of the triangulation. We compute the Laruent expansion of the red arc $\gamma^{(pq)}$ in the figure. The two snake graphs of the two loops ℓ_p and ℓ_q are shown in the top right of the figure. Note that the two snake graphs have the same shape, but not the same labels. The pictures at the bottom of the figure show the posets of γ-symmetric matchings for both snake graphs. The corresponding monomials $\overline{x}(P)\overline{y}(P)$ are listed below in the shape of the two posets.

$x_1x_3x_4x_5x_9\, y_1y_2y_3y_4y_5$	$x_1x_5x_7x_8x_9\, y_1y_6y_7y_8y_9$
$x_4x_5^2x_9x_{10}\, y_1y_3y_4y_5$	$x_5x_8x_9^2x_{11}\, y_1y_7y_8y_9$
$x_2x_4x_5x_6x_{10}\, y_3y_4y_5 \quad x_2x_5^2x_9x_{12}\, y_1y_4y_5$	$x_2x_6x_8x_9x_{11}\, y_7y_8y_9 \quad x_5x_6x_9^2x_{12}\, y_1y_8y_9$
$x_2^2x_5x_6x_{12}\, y_4y_5 \quad x_2x_3x_5x_9x_{11}\, y_1y_5$	$x_2x_6^2x_9x_{12}\, y_8y_9 \quad x_5x_6x_7x_9x_{10}\, y_1y_9$
$x_2^2x_3x_6x_{11}\, y_5$	$x_2x_6^2x_7x_{10}\, y_9$
$x_1x_2x_3x_4x_6$	$x_1x_2x_6x_7x_8$

We want to find all γ-compatible pairs. The four perfect matchings on the lower left side of the first poset all have horizontal edges on the first tile. These edges have labels 2 and 6. Therefore, each of these matchings forms a γ-compatible pair with each of the four matchings on the lower left side of the second poset. Similarly, the four perfect matchings on the upper right side of the first poset all have horizontal edges on the last tile. These edges have labels 5 and 9. Therefore, each of these matchings forms a γ-compatible pair with each of the four matchings on the upper right side of the second poset. There are no other γ-compatible pairs, so we have a total of 32 pairs. The Laurent polynomial for $x_{\gamma^{(pq)}}$ is shown in Figure 5.6.

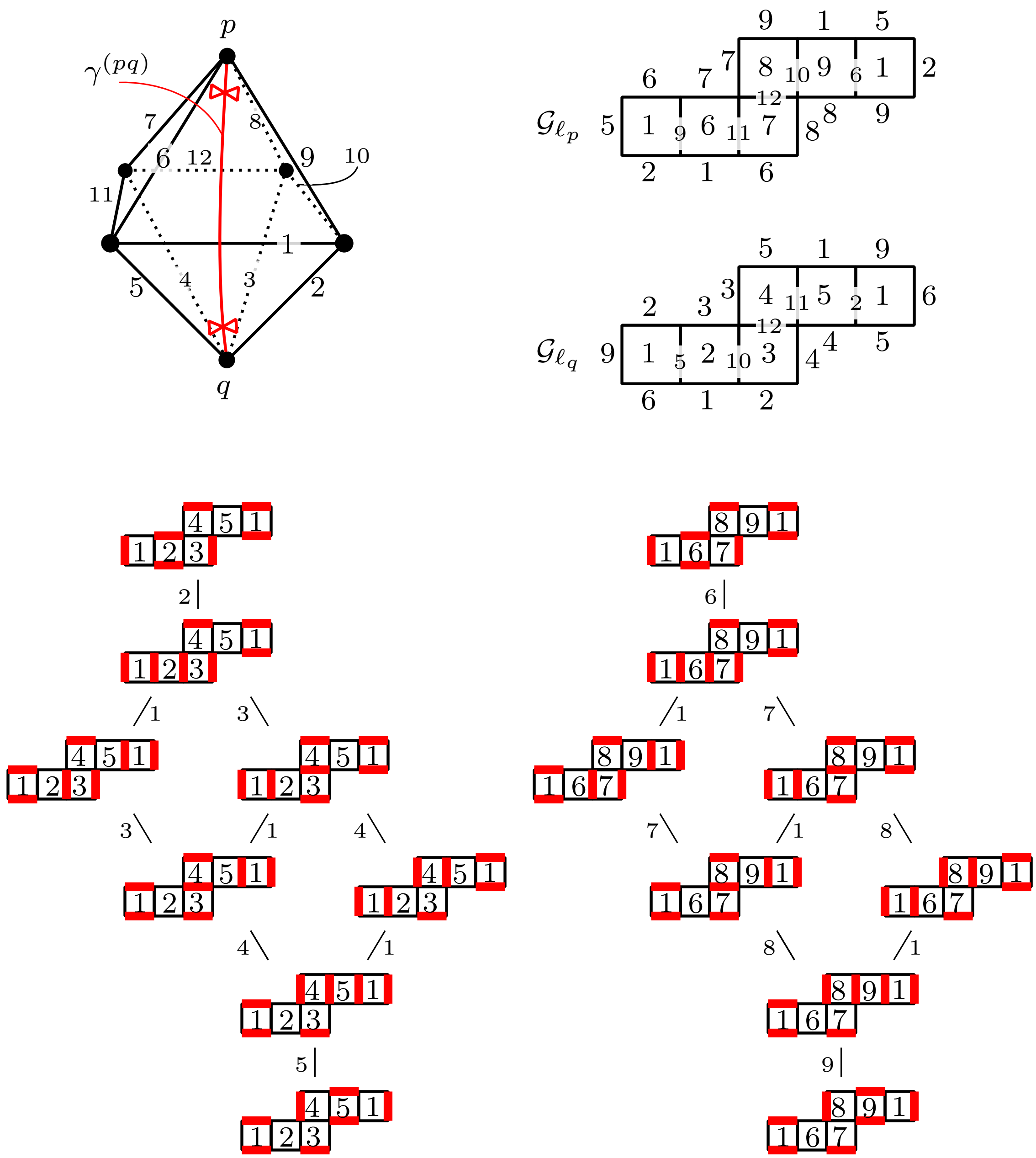


Figure 5.5: Ideal triangulation T° and doubly-notched arc γ_3.

$$
\frac{\begin{array}{c}
(x_1^2x_3x_4x_5x_7x_8x_9\,y_1y_2y_3y_4y_5y_6y_7y_8y_9\\
+x_1x_3x_4x_5x_8x_9^2x_{11}\,y_1y_2y_3y_4y_5y_7y_8y_9\\
+x_1x_3x_4x_5x_6x_9^2x_{12}\,y_1y_2y_3y_4y_5y_8y_9\\
+x_1x_3x_4x_5x_6x_7x_9x_{10}\,y_1y_2y_3y_4y_5y_9\\[1em]
+x_1x_4x_5^2x_7x_8x_9x_{10}\,y_1y_3y_4y_5y_6y_7y_8y_9\\
+x_4x_5^2x_8x_9^2x_{10}x_{11}\,y_1y_3y_4y_5y_7y_8y_9\\
+x_4x_5^2x_6x_9^2x_{10}x_{12}\,y_1y_3y_4y_5y_8y_9\\
+x_4x_5^2x_6x_7x_9x_{10}^2\,y_1y_3y_4y_5y_9\\[1em]
+x_1x_2x_5^2x_7x_8x_9x_{12}\,y_1y_4y_5y_6y_7y_8y_9\\
+x_2x_5^2x_8x_9^2x_{11}x_{12}\,y_1y_4y_5y_7y_8y_9\\
+x_2x_5^2x_6x_9^2x_{12}^2\,y_1y_4y_5y_8y_9\\
+x_2x_5^2x_6x_7x_9x_{10}x_{12}\,y_1y_4y_5y_9\\[1em]
+x_1x_2x_3x_5x_7x_8x_9x_{11}\,y_1y_5y_6y_7y_8y_9\\
+x_2x_3x_5x_8x_9^2x_{11}^2\,y_1y_5y_7y_8y_9\\
+x_2x_3x_5x_6x_9^2x_{11}x_{12}\,y_1y_5y_8y_9\\
+x_2x_3x_5x_6x_7x_9x_{10}x_{11}\,y_1y_5y_9\\[2em]
+x_2x_4x_5x_6x_8x_9x_{10}x_{11}\,y_3y_4y_5y_7y_8y_9\\
+x_2x_4x_5x_6^2x_9x_{10}x_{12}\,y_3y_4y_5y_8y_9\\
+x_2x_4x_5x_6^2x_7x_{10}^2\,y_3y_4y_5y_9\\
+x_1x_2x_4x_5x_6x_7x_8x_{10}\,y_3y_4y_5\\[1em]
+x_2^2x_5x_6x_8x_9x_{11}x_{12}\,y_4y_5y_7y_8y_9\\
+x_2^2x_5x_6^2x_9x_{12}^2\,y_4y_5y_8y_9\\
+x_2^2x_5x_6^2x_7x_{10}x_{12}\,y_4y_5y_9\\
+x_1x_2^2x_5x_6x_7x_8x_{12}\,y_4y_5\\[1em]
+x_2^2x_3x_6x_8x_9x_{11}^2\,y_5y_7y_8y_9\\
+x_2^2x_3x_6^2x_9x_{11}x_{12}\,y_5y_8y_9\\
+x_2^2x_3x_6^2x_7x_{10}x_{11}\,y_5y_9\\
+x_1x_2^2x_3x_6x_7x_8x_{11}\,y_5\\[1em]
+x_1x_2x_3x_4x_6x_8x_9x_{11}\,y_7y_8y_9\\
+x_1x_2x_3x_4x_6^2x_9x_{12}\,y_8y_9\\
+x_1x_2x_3x_4x_6^2x_7x_{10}\,y_9\\
+x_1^2x_2x_3x_4x_6x_7x_8)
\end{array}}{x_1x_2x_3x_4x_5x_6x_7x_8x_9}
$$

Figure 5.6: The Laurent polynomial of $x_{\gamma^{(pq)}}$.

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