# CRYSTALLOGRAPHIC ROOT 

## SYSTEMS

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## Abstract

This project shows how the attempt to classify all semisimple Lie algebras in an algebraically closed field of characteristic 0 led to the development of a rich area of study called root systems. Root systems have many applications, not only in Lie algebra, but also in geometry and combinatorics. We'll examine what is needed in order to classify all crystallographic root systems and will briefly touch on noncrystallographic ones as well. Additionally we will explain how this classification relates to semisimple Lie algebras and also take a slight detour midway to show that root systems are a powerful theory in their own right through the use of Coxeter groups.

A list of references used in this project can be found in the bibliography. Each section will be cited with a list of the one or two most dominantly used references for that section. No definitions are my own (unless otherwise stated) and come from the references as outlined in the section header. Each proof replicated from a source are reworded to fit better with the structure provided, and referenced accordingly to show the proof idea was not my original work.

## Chapter 1

## Introduction - Reflection Groups

### 1.1 Euclidean Space ${ }^{[6],[7]}$

Let $V$ be a nonempty set. We say that $V$ is a vector space over an arbitrary field $F$ if it is an abelian group under addition and for all $a, b \in F$ and $\nu, \mu \in V$ we have
(i) $a \nu \in V$
(ii) $a(\nu+\mu)=a \nu+a \mu$
(iii) $(a+b) \nu=a \nu+b \nu$
(iv) $a(b \nu)=(a b) \nu$
(v) $1 \nu=\nu$

We can see that $\mathbb{R}^{n}:=\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n}$ is a vector space over a field $F$ by letting the elements of $\mathbb{R}^{n}$ be ordered n-tuples $\left(\nu_{1}, \nu_{2}, \ldots, \nu_{n}\right)$ such that each $\nu_{i} \in F$ and by taking addition and scalar multiplication to be defined as:

$$
\begin{aligned}
\left(\nu_{1}, \nu_{2}, \ldots, \nu_{n}\right)+\left(\mu_{1}, \mu_{2}, \ldots, \mu_{n}\right) & =\left(\nu_{1}+\mu_{1}, \nu_{2}+\mu_{2}, \ldots, \nu_{n}+\mu_{n}\right) \\
a\left(\nu_{1}, \nu_{2}, \ldots, \nu_{n}\right) & =\left(a \nu_{1}, a \nu_{2}, \ldots, a \nu_{n}\right) \quad a \in F
\end{aligned}
$$

Definition 1.1.1. We say that a vector space $V$ over a field $F$ is an Euclidean space (or an inner product space) if we can define a positive definite symmetric bilinear form $(-,-)$ (called the inner product) on $V$. In other words for any $\nu, \mu, \omega \in V$ and $a, b \in F:$
(i) $(\nu, \nu) \geq 0$ and $(\nu, \nu)=0$ if and only if $\nu=0$
(ii) $(\nu, \mu)=(\mu, \nu)$
(iii) $(a \nu+b \omega, \mu)=a(\nu, \mu)+b(\omega, \mu)$
(positive definite)
(symmetric)
(bilinear)

As $(\nu, \nu)$ is always positive we can define the length of a vector as

$$
\|\nu\|=\sqrt{(\nu, \nu)}
$$

We also define the angle $\theta(\nu, \mu)$ between any two vectors $\nu$ and $\mu$ as

$$
\cos \theta(\nu, \mu)=\frac{(\nu, \mu)}{\|\nu\|\|\mu\|}
$$

To see that $\mathbb{R}^{n}$ is an Euclidean space we need an inner product on it. We thus define the dot product of any two vectors $\nu$ and $\mu$ in $\mathbb{R}^{n}$ as follows:

$$
(\nu, \mu) \mapsto \nu \cdot \mu=\nu_{1} \mu_{1}+\nu_{2} \mu_{2}+\ldots+\nu_{n} \mu_{n}
$$

It is easy to see the dot product is an inner product on $\mathbb{R}^{n}$ thus making $\mathbb{R}^{n}$ an Euclidean space. We call $\mathbb{R}^{n}$ the real Euclidean space. Similarly we can define an inner product on the complex numbers which allow us to call $\mathbb{C}^{n}$ the complex Euclidean space.

We start looking at me features this inner product allows us to have.

Definition. Any two vectors are orthogonal if $(\nu, \mu)=0$.

It's easy to notice that orthogonality implies that the angle $\theta$ between two vectors must be $\frac{\pi}{2}$. If we have a set of vectors one would expect multiple vectors to potentially be orthogonal to any given one motivating us to define the following.

Definition. If $W$ is a subspace of an Euclidean space $V$ then the orthogonal complement of $W$ is the set

$$
W^{\perp}:=\{\nu \in V \mid \quad(\nu, \omega)=0 \text { for all } \omega \in W\}
$$

We easily see that $W^{\perp}$ is a subspace of $V$. We next state an important theorem. A proof can be found in [6].

Theorem 1.1.2 (Kernel-image theorem). Let $V$ and $W$ be vector spaces over some field $F$ and $\phi$ be a linear map from $V$ to $W$. Then

$$
\operatorname{dim}(i m(\phi))+\operatorname{dim}(\operatorname{ker}(\phi))=\operatorname{dim}(V)
$$

By the Kernel-image theorem we see that an inner product allows us to create a map from $V$ to $W$ such that

$$
\operatorname{dim}(W)+\operatorname{dim}\left(W^{\perp}\right)=\operatorname{dim}(V)
$$

Definition 1.1.3. If we are given any vector space $V$ over a field $F$ and we take the set of all linear transformations from $V$ to $F$ then that set is known as the dual space of $V$. The dual space is denoted as $V^{\star}$ and is a vector space over $F$ with the following operations

$$
\begin{aligned}
(\phi+\psi)(\nu) & =\phi(\nu)+\psi(\nu) \\
(a \phi)(\nu) & =a(\phi(\nu))
\end{aligned}
$$

for all $\phi, \psi \in V^{\star}, \nu \in V$ and $a \in F$.
It can be shown that for finite dimensional vector spaces there is a basis of elements that generate the whole dual space. This basis is called the dual basis. It can also be shown that $\operatorname{dim} V^{\star}=\operatorname{dim} V$. We state without proof the following theorem which has a pleasant corollary. A proof can be found in [6].

Theorem 1.1.4. If $V$ and $W$ are vector spaces over a field $F$ such that $\operatorname{dim} V=\operatorname{dim} W$ then $V$ and $W$ are isomorphic, written $V \cong W$.

Corollary 1.1.5. $V \cong V^{\star}$

### 1.2 Reflection Groups ${ }^{[8],[9]}$

Definition 1.2.1 (Reflection). A reflection in real Euclidean space $V$ can be thought of as a linear transformation $s$ on V which sends a nonzero vector $\alpha$ to its negative and fixes point-wise the hyperplane $H_{\alpha}$ (a subspace with codimension 1).

The standard formula for describing a reflection is

$$
s_{\alpha}(\nu)=\nu-\frac{2(\nu, \alpha)}{(\alpha, \alpha)} \alpha .
$$

To see that our formula coincides with our definition we recall that two vectors are orthogonal if $(\nu, \mu)=0$. Let's suppose $\alpha$ is a nonzero vector we are sending to its negative, and $\eta$ is a nonzero vector in the hyperplane of $\alpha\left(\eta \in H_{\alpha}\right)$.

$$
\begin{aligned}
s_{\alpha}(\alpha) & =\alpha-\frac{2(\alpha, \alpha)}{(\alpha, \alpha)} \alpha & s_{\alpha}(\eta) & =\eta-\frac{2(\eta, \alpha)}{(\alpha, \alpha)} \alpha \\
& =\alpha-2 \alpha & & =\eta-\frac{0}{(\alpha, \alpha)} \alpha \\
& =-\alpha & & =\eta
\end{aligned}
$$

We see that all $\eta \in H_{\alpha}$ are fixed and all $\alpha$ are reflected to their negative. Thus our definition of reflection matches our formula for reflections. Notice also that for any nonzero $c \in \mathbb{R}$

$$
\begin{aligned}
s_{c \alpha}(\nu) & =\nu-\frac{2(\nu, c \alpha)}{(c \alpha, c \alpha)} c \alpha \\
& =\nu-\frac{2 c c(\nu, \alpha)}{c c(\alpha, \alpha)} \alpha \\
& =\nu-\frac{2(\nu, \alpha)}{(\alpha, \alpha)} \alpha \\
& =s_{\alpha}(\nu) .
\end{aligned}
$$

Therefore scaling $\alpha$ has no effect on the reflection. We also see that reflections preserve orthogonality as the inner product is symmetric and bilinear

$$
\begin{aligned}
\left(s_{\alpha}(\mu), s_{\alpha}(\nu)\right) & =\left(\mu-\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \alpha, \nu-\frac{2(\nu, \alpha)}{(\alpha, \alpha)} \alpha\right) \\
& =(\mu, \nu)-\left(\mu, \frac{2(\nu, \alpha)}{(\alpha, \alpha)} \alpha\right)-\left(\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \alpha, \nu\right)+\left(\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \alpha, \frac{2(\nu, \alpha)}{(\alpha, \alpha)} \alpha\right) \\
& =(\mu, \nu)-\frac{2(\nu, \alpha)}{(\alpha, \alpha)}(\mu, \alpha)-\frac{2(\mu, \alpha)}{(\alpha, \alpha)}(\alpha, \nu)+\frac{4(\mu, \alpha)(\nu, \alpha)}{(\alpha, \alpha)(\alpha, \alpha)}(\alpha, \alpha) \\
& =(\mu, \nu)-\frac{4(\mu, \alpha)(\nu, \alpha)}{(\alpha, \alpha)}+\frac{4(\mu, \alpha)(\nu, \alpha)}{(\alpha, \alpha)} \\
& =(\mu, \nu) .
\end{aligned}
$$

Finally note that $s_{\alpha}^{2}=1$ and therefore every reflection has order 2 .

Definition 1.2.2. A reflection group is a finite group generated by reflections.

We next describe some interesting reflection groups.


Figure 1.1: Reflections of $D_{4}$

### 1.2.1 $\quad I_{2}(m)$

The first reflection group we will look at is the dihedral group. The dihedral group of order $2 m$ consists of reflections and rotations of a regular polygon with $m$ labelled vertices. As an example if we take a square and label its vertices we can permute the labels by either reflecting or rotating the square.

In order to make the dihedral group into a reflection group we must convert all rotations into reflections so that only reflections remain in our group. In order to do this we first notice that a rotation can be made into a product of two reflections. Looking at figure 1.1, we see that in order to rotate a vector $\alpha_{1}$ to $\alpha_{2}$ we can compose two reflections by first taking a reflection $s_{\gamma}$ (over hyperplane $H_{\gamma}$ ) and then taking $s_{\beta}$ (over hyperplane $H_{\beta}$ ). From here we can easily see the following:

Proposition 1.2.3. Let $D_{m}$ be the dihedral group of order $2 m$. Over $D_{m}$ reflections form two conjugacy classes when $m$ is even and one conjugacy class when $m$ is odd. Proof. We first claim that if $\phi$ is the angle between two vectors $\alpha$ and $\beta$ then the reflection $s_{\beta} \circ s_{\alpha}$ rotates a vector by $2 \phi$. Indeed, let $s_{\alpha}$ and $s_{\beta}$ be any two reflections, $\nu$ be any arbitrary vector in an Euclidean space $V$ and let $\phi$ be the angle between two vectors $\alpha$ and $\beta$. Any reflection will take a vector $\nu$ and rotate the vector to two times the angle between the vector and the hyperplane of that reflection. Therefore we see that the angle between $\nu$ and $\left(s_{\beta} \circ s_{\alpha}\right)(\nu)$ will be $\theta_{\alpha}+\theta_{\beta}$ where $\theta_{i}$ is the angle that the vector is rotated and is twice the angle between the vector and the hyperplane $H_{i}$ (for $i \in\{\alpha, \beta\}$ ). As $\phi$ is the angle between $\alpha$ and $\beta$ we notice that

$$
\frac{\theta_{\beta}}{2}=\frac{\theta_{\alpha}}{2}-\phi
$$

and thus we see that $2 \phi=\theta_{\beta}+\theta_{\alpha}$. Also, as a reflection reverses the order of the


Figure 1.2: Reflection composition
vertices we see that two permutations keeps the vertices in their original order. Thus our claim is proved. We can see an example in figure 1.2. As a rotation can be considered to be two reflections we can look at the conjugacy classes of reflections based off of rotations. We note without proof that the dihedral group can be seen to be $D_{m}=\left\{\mu^{i} \rho^{j} \mid i \in\{0, \ldots, m-1\}, j \in\{0,1\}\right\}$ where $\mu$ is a rotation and $\rho$ is any fixed reflection. Therefore we look at conjugacy classes generated by the above group.

When $m$ is odd we see that rotations form the whole group and therefore only a single conjugacy class exists. When $m$ is even we notice that the rotations split into 2 classes: those with even power and those with odd power. We can thus create two conjugacy classes (one of even power rotations and one of odd) and as rotations can just be thought of as reflections this completes the proof.

Therefore the dihedral group is just a group generated by reflections.
Reflection groups that have this construction are known as type $I_{2}(m)$ reflection groups where $m \geq 3$. These groups have order $2 m$ and are generated by $m$ reflections.

### 1.2.2 $A_{n}$

We saw that for the dihedral group the reflections permuted the labelled vertices. We thus ask whether we can make the symmetric group $S_{n+1}$ (the group of permutations of $n+1$ elements) into a reflection group. We can look at $S_{n+1}$ as a subgroup of $O(n+1, \mathbb{R})(n+1 \times n+1$ orthogonal matrices with entries in $\mathbb{R})$. Let our standard basis for $O(n+1, \mathbb{R})$ be the vectors $\varepsilon_{i}$ where $\varepsilon_{i}$ is an $n+1$ length column vector with all 0 s and a 1 in the $i$ th row. We can let a permutation in $S_{n+1}$ act on $O(n+1, \mathbb{R})$ by permuting the subscripts of the standard basis. For example let $\nu=\left(a_{1} \varepsilon_{1}+a_{2} \varepsilon_{2}+\ldots a_{n} \varepsilon_{n}\right)$ and $\sigma \in S_{n+1}$ where $a_{i} \in \mathbb{R}$, then $s_{\sigma}(\nu)=\left(a_{1} \varepsilon_{\sigma(1)}+a_{2} \varepsilon_{\sigma(2)}+\ldots+a_{n} \varepsilon_{\sigma(n)}\right)$.

In this way we notice that a reflection can be thought of as a transposition $(i, j)$ which sends $\varepsilon_{i}-\varepsilon_{j}$ to its negative. This reflection would fix all vectors whose $i$ th and $j$ th component are equal as then $a_{i}-a_{j}=-\left(a_{i}-a_{j}\right)=0$ (where $a_{i}, a_{j}$ are the $i$ th and $j$ th component in our vector). As any permutation can be thought of as a product of transpositions we can see that $S_{n+1}$ can be viewed as a reflection group generated by the transpositions $(i, i+1)(1 \leq i \leq n)$, pending that a transposition is the only possible reflection.

Proposition 1.2.4. Transpositions are the only reflections in $S_{n+1}$ when $S_{n+1}$ is viewed as a subgroup of $O(n+1, \mathbb{R})$
Proof. Suppose we have some other type of reflection in $S_{n+1}$. Let $\rho$ be a reflection of this type such that $\rho$ sends the first $m$ basis elements to their negative. (If not we can reorder our basis elements). Let $\nu$ be a vector in $\mathbb{R}^{n+1}$ that gets sent to its negative. As $\nu$ is a linear combination of elements of the standard basis and since $\rho$ would only alter the first $m<n+1$ subscripts we see that

$$
\rho(\nu)=-\nu=\sum_{i=1}^{m}-a_{i} \varepsilon_{i} \quad a_{i} \in \mathbb{R}
$$

such that

$$
\nu=\sum_{i=1}^{m} a_{i} \varepsilon_{i} \quad a_{i} \in \mathbb{R} .
$$

We notice that our summations are only up to $m$ as $\rho$ doesn't alter any other standard basis element. Now let's look at a vector $\mu=\sum_{i=1}^{m} b_{i} \varepsilon_{i}$ that is fixed under $\rho$. $\rho$ sends each $\varepsilon_{i}$ to its negative for $i \leq m$. The only way this can happen is if $b_{i}=-\sum b_{k}$ for some subset of subscripts $k<m$. We see that case is the same thing as saying we first send $b_{i}-b_{k_{1}}$ to its negative, then $b_{i}-b_{k_{2}}$ to its negative and continue until we get to our equality. But then these reflections are just the transpositions we described and therefore we have a contradiction.

Looking at the reflection group with generators $(i, i+1), 1 \leq i \leq n$ we see that the only fixed points in $\mathbb{R}^{n+1}$ are the vectors spanned by the standard basis with all equal coefficients

$$
a\left(\varepsilon_{1}+\varepsilon_{2}+\ldots+\varepsilon_{n+1}\right) \quad a \in \mathbb{R}
$$

and that the only vectors that get sent to their negatives are the vectors whose coefficients of the standard basis all sum up to 0

$$
\nu=\sum_{i=1}^{n+1} a_{i} \varepsilon_{i} \quad \text { such that } \quad \sum_{i=1}^{n+1} a_{i}=0
$$

We thus see that $S_{n+1}$ acting on an $n$ dimensional Euclidean space $V$ in the way just described fixes only the origin. We say that a reflection group $W$ (short for Weyl group) is essential if all the reflections in $W$ span the whole set of orthogonal vectors of $V$ or in other words $W$ fixes no nonzero points in $V$.

Reflection groups that have this construction are known as type $A_{n}$ reflection groups where $n \geq 1$. These groups have order $n$ ! and are generated by $n(n-1)$ reflections.

### 1.2.3 $B_{n}$

Continuing with our discussion of reflection groups for $\mathbb{R}^{n}$, we can expand our set of reflections by considering other reflections in $\mathbb{R}^{n}$ that are not permutations (and thus not in $S_{n}$ ). For this we look at reflections that send $\varepsilon_{i}$ to its negative and fix all other basis elements.

By looking at sign changes we notice that we can generate a group of order $2^{n}$ as we have $n$ elements in our standard basis and each element can either be positive or negative. Therefore this group is isomorphic to $(\mathbb{Z} / 2 \mathbb{Z})^{n}$ and thus we can create a new reflection group $W$ that is the semidirect product of our group of sign changes and of the permutation group ( $W=(\mathbb{Z} / 2 \mathbb{Z})^{n} \rtimes S_{n}$ ). It's fairly obvious that this group would have order $2^{n} n!$ as its intersection with $S_{n}$ would be trivial. And since $S_{n}$ spans the whole set of orthogonal vectors in $\mathbb{R}^{n}$ we therefore have that $W$ must also and thus $W$ is essential relative to $\mathbb{R}^{n}$.

We call reflection groups of this construction as having type $B_{n}$ where $n \geq 2$. These groups have order $2^{n} n!$ and are generated by $2 n^{2}$ reflections.

## $1.2 .4 D_{n}$

With reflection groups of type $B_{n}$ we saw that introducing sign changes generated a new reflection group. If we restrict our sign changes so that we alter two elements at
a time $\left(\varepsilon_{i}+\varepsilon_{j} \mapsto-\left(\varepsilon_{i}+\varepsilon_{j}\right), i \neq j\right)$. We see that this generates a group of order $2^{n-1}$ isomorphic to $(\mathbb{Z} / 2 \mathbb{Z})^{n-1}$.

As before if we take the semidirect product of the permutation group and this group we get a new reflection group $W=(\mathbb{Z} / 2 \mathbb{Z})^{n-1} \rtimes S_{n}$ which by the same argument as for $B_{n}$ is essential relative to $\mathbb{R}^{n}$.

Reflection groups of this construction are known as being of type $D_{n}$ where $n \geq 4$. Also we see this group has order $2^{n-1} n$ ! and is generated by $2 n^{2}-2 n$ reflections.

## Chapter 2

## Lie Algebras

Having a few examples under our belt we now turn our attention to Lie algebras. Lie algebra has very close ties with reflection groups and the construction of root systems. In fact it will turn out that all simple Lie algebras have an associated irreducible root system. It will also turn out that the simple Lie algebras give rise to all crystallographic root systems.

### 2.1 History ${ }^{[2],[11]}$

Lie algebras came to being due to the efforts of the Norwegian Sophus Lie (1842 to 1899). Lie began studying infinitesimal transformation groups and continuous groups with the hopes of developing a theory much like that developed by Galois. His interests led him to what he called the infinitesimal group, although we now call it a Lie algebra. His seminal work was produced with his assistant Engel, and was released in a treatise called Theorie der Transformationsgruppen from 1888-1893. Later Killing, Cartan, Engel, Levi, Malcev, Weyl and Dynkin would help the field of Lie algebra realise its true potential. Killing introduced the characteristic equation to Lie algebras. Engel produced what is now known as Engel's theorem and Cartan introduced what we now call the Killing form and conjectured that every Lie algebra is the sum of its radical and a semisimple subalgebra (which we now call a Cartan subalgebra in honour). Levi proved this conjecture valid and Malcev then showed its uniqueness. Weyl then showed that linear representations of semisimple Lie algebras were completely reducible and also first introduced the term Lie algebra. He was the first to
start calling these infinitesimal transformation groups as Lie groups in honour of Lie's work and contributions. Finally, Dynkin finished by classifying in a graph theoretic manner all semisimple Lie algebras.

### 2.2 Definitions ${ }^{[5],[8]}$

Definition 2.2.1 (Lie Algebra). A Lie algebra $L$ over a field $F$ is a vector space with a bilinear operation called the bracket or commutator which takes $L \times L \rightarrow L$ by the map $(x, y) \mapsto[x, y]$, such that:
(L1) $[x, x]=0$ for all $x$ in $L$
(L2) $[x,[y, z]]+[y,[z, x]]+[z,[x, y]]=0$ for $x, y, z \in L \quad$ called the Jacobi Identity

We quickly notice that by bilinearity and (L1) we can show anticommutativity:

$$
\begin{aligned}
0 & =[x+y, x+y] \\
& =[x, x]+[x, y]+[y, x]+[y, y] \\
& =[x, y]+[y, x] \\
\therefore & \Rightarrow-[y, x]=[x, y]
\end{aligned}
$$

Note that when the characteristic of the field is not 2, anticommutativity implies (L1). This fails in characteristic of field 2 as $[x+y, x+y]=[x, y]+[y, x]=[x, y]-[x, y]=$ $[x, y]+[x, y]=2[x, y]$ allowing $[x, y]$ to be anything.

Letting $V$ be a finite dimensional vector space over a field $F$ and $\operatorname{End}(V)$ be the group of endomorphisms of $V$. Then by defining the bracket for $x, y \in \operatorname{End}(V)$ as $[x, y]=x \circ y-y \circ x, \operatorname{End}(V)$ becomes a Lie algebra over $F$. Let this Lie algebra of endomorphisms be denoted $\mathfrak{g l}(V)$ and call it the general linear algebra as it's closely related to the general linear group $G L(V)$. A subalgebra of $\mathfrak{g l}(V)$ is called a linear Lie algebra.

An ideal $I$ of a Lie algebra $L$ is a subspace of $L$ such that $[L, I] \subseteq I$ where $[L, I]:=\{[x, y] \mid x \in L, y \in I\}$. A Lie algebra $L$ is abelian if $[L, L]=\{0\}([L, L]$ is called the derived subalebra of $L$ ). A Lie algebra $L$ is simple if its only ideals are $L$ and $\{0\}$ and $L$ is non-abelian.

If $I$ is an ideal of $L$ over a field $F$ then we can construct the factor algebra $L / I$ in the same way as a factor ring. We define all operations as follows

$$
\begin{aligned}
\text { addition } & (x+I)+(y+I)=(x+y)+I \\
\text { scalar multiplication } & a(x+I)=(a x)+I \\
\text { multiplication } & {[x+I, y+I]=[x, y]+I }
\end{aligned}
$$

where $a \in F$ and $x, y \in L$.
The center of an algebra is defined as $Z(L):=\{z \in L \mid[x, z]=0 \forall x \in L\}$. The centraliser of a subalgebra $\mathbf{X}$ is defined as $C_{L}(X):=\{c \in L \mid[c, x]=0 \forall x \in X\}$. The normaliser of a subalgebra $\mathbf{X}$ is defined as $N_{L}(X):=\{n \in L \mid[n, x] \in X \forall x \in$ $X\}$. A subalgebra is said to be self-normalising if $X=N_{L}(X)$.

As with any algebraic structure, Lie algebras have morphisms. A Lie algebra homomorphism is defined as a map $\phi: L \rightarrow L^{\prime}$ such that $\phi([x, y])=[\phi(x), \phi(y)]$ for all $x, y \in L$. We define monomorphism to be an injective homomorphism (or equivalently if $\operatorname{ker}(\phi)=\{0\}$ ), an epimorphism to be a surjective homomorphism, an isomorphism to be a bijective homomorphism, an endomorphism to be a homomorphism from $L$ to itself, and an automorphism to be a bijective endomorphism. We state the three isomorphism theorems without proof.

Theorem 2.2.2 (Isomorphism Theorems). Let L and $L^{\prime}$ be Lie algebras over a field $F, \phi$ be a Lie homomorphism, and I and $J$ be ideals of $L$ such that $I \subseteq J$. Then the following are true:
(i) $L / \operatorname{ker}(\phi) \cong i m(\phi)$
(ii) $(I+J) / J \cong I /(I \cap J)$
(iii) $(L / I) /(J / I) \cong L / J$

### 2.3 Solvability and Nilpotency ${ }^{[8],[13]}$

By looking at the ideals of a Lie algebra we can tell a lot about the structure of the Lie algebra itself.

Definition 2.3.1. The lower central series of a Lie algebra $L$ is a sequence $L=$ $L^{1} \supseteq L^{2} \supseteq \ldots \supseteq L^{n} \supseteq \ldots$ such that each $L^{i}$ is recursively defined by:

$$
\begin{aligned}
L^{1} & =L \\
L^{n+1} & =\left[L, L^{n}\right] \quad \forall n \geq 1 .
\end{aligned}
$$

If there is a positive integer $N$ such that $L^{N}=\{0\}$ then we say $L$ is nilpotent. The least $N$ such that $L^{N}=\{0\}$ is known as the class of nilpotency of L .

Definition 2.3.2. The derived series of a Lie algebra $L$ is a sequence $L=L^{(0)} \supseteq$ $L^{(1)} \supseteq \ldots \supseteq L^{(n)} \supseteq \ldots$ such that each $L^{(i)}$ is recursively defined by:

$$
\begin{aligned}
L^{(0)} & =L \\
L^{(n+1)} & =\left[L^{(n)}, L^{(n)}\right] \quad \forall n \geq 0 .
\end{aligned}
$$

If there is a positive integer $N$ such that $L^{(N)}=\{0\}$ then we say $L$ is solvable.

Theorem 2.3.3 (On solvable and nilpotent Lie algebras). Let $L$ and $M$ be Lie algebras and $\phi$ be a Lie homomorphism such that $\phi: L \rightarrow M$
(i) If $L$ is solvable (resp. nilpotent) then all subalgebras of $L$ are solvable (resp. nilpotent).
(ii) If $L$ is solvable (resp. nilpotent) then $\phi(L)$ is solvable (resp. nilpotent).
(iii) The direct sum of solvable (resp. nilpotent) algebras is solvable (resp. nilpotent)
(iv) If $I$ is a solvable ideal of $L$ and $L / I$ is solvable then $L$ is solvable.
(v) If $I, J$ are solvable ideals of $L$, then so is $I+J$.
(vi) If $L$ is nilpotent and nonzero, then $Z(L) \neq\{0\}$.
(vii) If $L / Z(L)$ is nilpotent, then $L$ is nilpotent.

Proof. Proofs for all parts not proved below can be found in [8].
(iii) Suppose that $L=L_{1} \oplus L_{2}$ such that $L_{1}$ and $L_{2}$ are solvable (resp. nilpotent).

We want to first show that $L^{(i)}=L_{1}^{(i)} \oplus L_{2}^{(i)}$ (resp. $L^{i}=L_{1}^{i} \oplus L_{2}^{i}$ )
Let $n=0$ (resp. 1). Thus $L^{1}=L^{(0)}=L=L_{1} \oplus L_{2}=L_{1}^{(0)} \oplus L_{2}^{(0)}=L_{1}^{1} \oplus L_{2}^{1}$. Thus it is trivially true for $n=0$.

Let $L^{(n)}=L_{1}^{(n)} \oplus L_{2}^{(n)}$ (resp. $L^{n}=L_{1}^{n} \oplus L_{2}^{n}$ ) be true by induction hypothesis. Then by bilinearity of the commutator and as $L_{i} \cap L_{j}=\{0\}$, by definition of direct sum for $i \neq j$, we see

$$
\begin{aligned}
L^{(n+1)} & =\left[L^{(n)}, L^{(n)}\right] \\
& =\left[L_{1}^{(n)} \oplus L_{2}^{(n)}, L_{1}^{(n)} \oplus L_{2}^{(n)}\right] \\
& =\left[L_{1}^{(n)}, L_{1}^{(n)}\right] \oplus\left[L_{2}^{(n)}, L_{1}^{(n)}\right] \oplus\left[L_{1}^{(n)}, L_{2}^{(n)}\right] \oplus\left[L_{2}^{(n)}, L_{2}^{(n)}\right] \\
& =L_{1}^{(n+1)} \oplus L_{2}^{(n+1)} .
\end{aligned}
$$

Similarly for nilpotency we see that

$$
\begin{aligned}
L^{n+1} & =\left[L, L^{n}\right] \\
& =\left[L_{1} \oplus L_{2}, L_{1}^{n} \oplus L_{2}^{n}\right] \\
& =\left[L_{1}, L_{1}^{n}\right] \oplus\left[L_{2}, L_{1}^{n}\right] \oplus\left[L_{1}, L_{2}^{n}\right] \oplus\left[L_{2}, L_{2}^{n}\right] \\
& =L_{1}^{n+1} \oplus L_{2}^{n+1} .
\end{aligned}
$$

Thus if $L_{1}$ and $L_{2}$ are solvable (resp. nilpotent) we see that there exists positive integers $N_{1}$ and $N_{2}$ such that $L_{1}^{\left(N_{1}\right)}=\{0\}$ and $L_{2}^{\left(N_{2}\right)}=\{0\}$ (resp. $L_{1}^{N_{1}}=\{0\}$ and $L_{2}^{N_{2}}=\{0\}$ ). Thus if we take $N$ to be the greater of $N_{1}$ and $N_{2}$ we see that $L^{(N)}=L_{1}^{(N)} \oplus L_{2}^{(N)}=\{0\} \oplus\{0\}=\{0\}\left(\right.$ resp. $\left.L^{N}=L_{1}^{N} \oplus L_{2}^{N}=\{0\} \oplus\{0\}=\{0\}\right)$ thus $L$ is solvable (resp. nilpotent).

This can easily be extended by induction to be any number of direct sums, hence our proposition.

We can easily see that by the definitions of ideal and solvability we can create a maximum solvable ideal (take the direct sum of all proper solvable ideals). We call this maximum solvable ideal of a Lie algebra $L$ the radical of $L$ and denote it by $\operatorname{rad}(L)$. By the same procedure we can create a maximum nilpotent ideal and call it the nilradical of $L$. Whenever $\operatorname{rad}(L)=\{0\}$ we say that $L$ is semisimple. It is easily seen that any Lie algebra has a semisimple subalgebra.

Proposition 2.3.4. Let $L$ be a finite dimensional Lie algebra and $R=\operatorname{rad}(L)$, the radical of $L$. Then $L / R$ is semisimple.

Proof. (Outlined in [13]) If $L$ is semisimple then $R=\{0\}$ and $L / R$ is trivially semisimple. Therefore assume that $L$ is not semisimple and thus $R \neq\{0\}$. We want to show that $L / R$ has no non-trivial solvable ideals.

Suppose contrarily that $S / R$ is a non-trivial solvable ideal of $L / R$ where $S$ is an ideal of $L$ larger than $R$. As $R$ is the radical of $L$ it must also be the radical of $S$ (else $S$ would be the radical). But then we see that as $S / R$ and $R$ are solvable then by our previous proposition part iv $S$ is solvable as well. Thus $S$ is also a solvable ideal of $L$ that is larger than $R$ which is a contradiction.

### 2.4 Representations ${ }^{[5],[8]}$

We next introduce representions of Lie algebras. The main representation we use is called the adjoint representation which is the map $a d: L \rightarrow \mathfrak{g l}(L)$ where $x \mapsto a d x$. The map $a d$ allows us to construct the adjoint endomorphism $a d x: L \rightarrow L$ such that $(a d x)(y) \mapsto[x, y]$.

Proposition. ad is a Lie homomorphism.
Proof.

$$
\begin{aligned}
(\operatorname{ad}[x, y])(z) & =[[x, y], z] \\
& =[x,[y, z]]+[y,[z, x]] \\
& =[x,[y, z]]-[y,[x, z]] \\
& =(\operatorname{ad} x \circ \operatorname{ad} y)(z)-(\operatorname{ad} y \circ \operatorname{ad} x)(z) \\
& =[\operatorname{ad} x, \operatorname{ad} y](z)
\end{aligned}
$$

Sometimes we will use $a d_{L}$ when we want to emphasise which Lie algebra the adjoint representation is associated with.

We can also easily construct a matrix representation of $a d x$ by looking at the coefficients of each element in the algebra.

Example. As an example of the creation of an adjoint representation, let's demonstrate the above concepts with a 3 dimensional Lie algebra where our three basis
elements are

$$
e=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \quad h=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \quad f=\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right) .
$$

Defining the bracket as $[A, B]=A \cdot B-B \cdot A$ where $\cdot$ is matrix multiplication we easily see that

$$
[e, h]=-2 e \quad[e, f]=h \quad[f, h]=2 f
$$

Therefore with regards to the ordered basis $\{e, h, f\}$ we see that the adjoint representations are

$$
a d e=\left(\begin{array}{ccc}
0 & -2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right) \quad a d h=\left(\begin{array}{ccc}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{array}\right) \quad a d f=\left(\begin{array}{ccc}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 2 & 0
\end{array}\right) .
$$

This Lie algebra is normally denoted as $\mathfrak{s l}(2, F)$ and is isomorphic to $A_{1}$. It is the Lie algebra of linear matrices with trace equal to 0 .

We state without proof Ado's and Iwasawa's theorems (the proof may be found in [10])

Theorem 2.4.1 (Ado-Iwasawa Theorem). Let $L$ be a finite dimensional Lie algebra over a field $F$. Then $L$ has a finite dimensional representation which is injective.

This theorem was solved in the case when the characteristic of $F$ is 0 by Ado and when the characteristic of $F$ is a prime $p$ by Iwasawa. This gives us the nice corollary that every finite dimensional Lie algebra is isomorphic to a subgroup of $\mathfrak{g l}(L)$.

As every finite dimensional Lie algebra has a representation we can define the trace of an endomorphism in $\mathfrak{g l}(L)$ to be the trace of its representation in $G L(n, F)$ denoting the $n \times n$ matrices over a field $F$.

We end this subsection by stating the following two theorems without proof. Proofs can be found in [8].

Theorem 2.4.2 (Engel's Theorem). A Lie algebra $L$ is nilpotent if and only if for all $x \in L$, ad $x$ is nilpotent.

Theorem 2.4.3. $L$ is solvable if and only if $[L, L]$ is nilpotent.

### 2.5 Killing Form ${ }^{[5],[8]}$

Note. The rest of this project will assume that all fields are algebraically closed with characteristic 0 unless otherwise stated.

Definition 2.5.1 (Killing Form). The Killing form is a symmetric bilinear form on $L$ defined as:

$$
\kappa(x, y):=\operatorname{tr}(a d x \circ a d y)
$$

As trace is a commutative linear operation and composition is associative we see that the Killing form must be associative as well.

Proposition. The Killing form is associative.
Proof. We want to show that for all $x, y, z \in L, \kappa([x, y], z)=\kappa(x,[y, z])$.

$$
\begin{aligned}
\kappa([x, y], z) & =\operatorname{tr}(a d[x, y] \circ a d z) \\
& =\operatorname{tr}([a d x, a d y] \circ a d z) \\
& =\operatorname{tr}((a d x \circ a d y-a d y \circ a d x) \circ a d z) \\
& =\operatorname{tr}((a d x \circ a d y) \circ a d z)-\operatorname{tr}((a d y \circ a d x) \circ a d z) \\
& =\operatorname{tr}(a d x \circ(a d y \circ a d z))-\operatorname{tr}(a d y \circ(a d x \circ a d z)) \\
& =\operatorname{tr}(a d x \circ(a d y \circ a d z))-\operatorname{tr}((a d x \circ a d z) \circ a d y) \\
& =\operatorname{tr}(a d x \circ(a d y \circ a d z-a d z \circ a d y)) \\
& =\operatorname{tr}(a d x \circ[a d y, a d z]) \\
& =\operatorname{tr}(a d x \circ a d[y, z]) \\
& =\kappa(x,[y, z])
\end{aligned}
$$

Definition. Let $\kappa$ be the Killing form of a Lie algebra $L$. By above we see that $\kappa$ is a positive definite symmetric bilinear form and thus it makes sense for us to look at $L^{\perp}$ relative to $\kappa$. Thus for any subalgebra $S$ of $L$ we have $S^{\perp}:=\{x \in L \mid \kappa(x, y)=$ $0 \forall y \in S\}$. We say that $\kappa$ is nondegenerate if $L^{\perp}=\{0\} . L^{\perp}$ is known as the radical of $\kappa$.

Proposition 2.5.2. If $I$ is an ideal of a Lie algebra $L$ then $I^{\perp}$ is an ideal as well. Moreover,

$$
\operatorname{dim} I+\operatorname{dim} I^{\perp}=\operatorname{dim} L
$$

Proof. By definition, for all $x \in I^{\perp}$ and $y \in I$ we have that $\kappa(x, y)=0$. Let $a$ be an arbitrary element of $L . I^{\perp}$ is an ideal of $L$ if $[a, x] \in I^{\perp} .[a, x] \in I^{\perp}$ if $\kappa([a, x], y)=0$. We already know that as $x \in I^{\perp}$ we have that $\kappa(x, y)=\operatorname{tr}(a d x \circ a d y)=\operatorname{tr}(a d y \circ a d x)=$ 0 and by associativity of $\kappa$ we have that

$$
\begin{aligned}
\kappa([a, x], y) & =\kappa(a,[x, y]) \\
& =\operatorname{tr}(a d a \circ a d[x, y]) \\
& =\operatorname{tr}(a d a \circ(a d x \circ a d y-a d y \circ a d x)) \\
& =\operatorname{tr}(a d a \circ(a d x \circ a d y))-\operatorname{tr}(a d a \circ(a d y \circ a d x)) \\
& =0 .
\end{aligned}
$$

Thus $[a, x] \in I^{\perp}$ and therefore $I^{\perp}$ is an ideal.
Now by the Kernel-image theorem we know that $I$ and $I^{\perp}$ can be seen as the image and kernel respectively of a map which sends $I$ to $I$ and all other elements to $I^{\perp}$ by looking at the Killing form. Thus we easily get that $\operatorname{dim} I+\operatorname{dim} I^{\perp}=\operatorname{dim} L$.

An easy way to see if a Lie algebra is solvable is to look at its derived series. We state Cartan's criterion without proof.

Theorem 2.5.3 (Cartan's Criterion). Let $L$ be a linear Lie subalgebra of $\mathfrak{g l}(V)$ where $V$ is a finite dimensional vector space. If $\operatorname{tr}(x y)=0$ for all $x \in[L, L]$ and $y \in L$ then $L$ is solvable.

Equivalently, we could have stated the criterion as $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y)=0$ for all $x \in[L, L]$ and $y \in L$ implies that $L$ is solvable.

We state the following two theorems without proof. A detailed proof can be found in [5].

Theorem 2.5.4. A Lie algebra $L$ is semisimple if and only if its Killing form is nondegenerate.

Theorem 2.5.5. $L$ is a semisimple Lie algebra if and only if there exist unique simple ideals $L_{i}$ such that

$$
L=L_{1} \oplus L_{2} \oplus \ldots \oplus L_{n}
$$

### 2.6 Cartan Decompositions ${ }^{[8]}$

As we saw in the last section we can decompose a semisimple Lie algebra into unique simple ideals. We now turn to Cartan decompositions to see if we can get a nice way of representing those simple ideals. We go through the method of toral subalgebras as presented by Humphreys in [8].

The first thing to note is that any element $x$ in a Lie algebra $\operatorname{End}(V)$ is semisimple if its eigenvalues over the field $F$ are all distinct. Another way of saying this is that $x$ must be diagonalisable. We also note that any element $x$ in $\operatorname{End}(V)$ can be decomposed into semisimple and nilpotent parts. Thus we say that $x=x_{s}+x_{n}$ where $x_{s}$ is the semisimple part and $x_{n}$ is the nilpotent part of $x$. This decomposition is known as the Jordan-Chevalley decomposition. We state without proof the following proposition. The proof can be found in [8].

Proposition 2.6.1. Let $V$ be a finite dimensional vector space over an algebraically closed field $F$ (of arbitrary characterisitic). Let $x \in \operatorname{End}(V)$. Then the following hold:
(i) There exist $x_{s}, x_{n} \in \operatorname{End}(V)$ such that $x=x_{s}+x_{n}$ is the Jordan-Chevalley decomposition and $x_{s}$ and $x_{n}$ commute.
(ii) $x_{s}$ and $x_{n}$ commute with any endomorphism which commutes with $x$.
(iii) If $A \subset B \subset V$ are all subspaces, and $x: B \rightarrow A$, then $x_{s}, x_{n}: B \rightarrow A$.
(iv) If $(a d x)=(a d x)_{s}+(a d x)_{n}$ and the characteristic of $F$ is 0 then $(a d x)_{s}=a d x_{s}$ and $(a d x)_{n}=a d x_{n}$

Definition 2.6.2 (Toral Subalgebra). Let $L$ be a nonzero semisimple Lie algebra such that $L$ is not nilpotent. A toral subalgebra is a subalgebra consisting of the span of semisimple parts of the Jordan-Chevalley decomposition of elements of a Lie algebra $L$.

It can easily be seen that toral subalgebras are abelian which we state here as a lemma.

Lemma 2.6.3. A toral subalgebra of semisimple Lie algebra $L$ is abelian.

As toral subalgebras are abelian, we can define a maximal toral subalgebra $H$ of $L$ such that $H$ is not contained in any other toral subalgebra. Now as $H$ is abelian we see that we can look at the adjoint endomorphisms of the elements of $H$. We see that $a d_{L} H$ consists of the commutative semisimple endomorphisms of $L$, which by a readily available result in linear algebra shows that $a d_{L} H$ is simultaneously diagonalisable. This allows us to define the following definitions.

Definition 2.6.4. Let $L$ be semisimple Lie algebra in an algebraically closed field $F$. As $a d_{L} H$ is simultaneously diagonalisable $L$ is the direct sum of subspaces $L_{\alpha}:=\{x \in$ $L \mid[h, x]=\alpha(h) x \forall h \in H\}$ where the $\alpha$ ranges over $H^{\star}$. We let $\Phi$ be the set of $\alpha$ such that $L_{\alpha} \neq\{0\}$. The elements of $\Phi$ are known as roots of $L$ (relative to $H$ ) and each $L_{\alpha}$ where $\alpha$ is a root is called the root space. With this notation we can thus rewrite our decomposition to be

$$
\begin{equation*}
L=L_{0} \oplus \bigoplus_{\alpha \in \Phi} L_{\alpha} \tag{2.1}
\end{equation*}
$$

and call it the root space decomposition or the Cartan decomposition. Note that $L_{0}$ just consists of all central elements and thus $L_{0}=C_{L}(H)$.

Our next aim is to show that $H$ is actually equal to $C_{L} H$. We 'prove' this by outlining the proof given by Humphreys in [8].

Theorem 2.6.5. $H=C_{L}(H)$

Proofs will not be given to any propositions unless not found in [8]. In all propositions that follow, $H$ is the maximal toral subalgebra of a semisimple Lie algebra $L$ over an algebraically closed field $F$ of characteristic 0 .

Proposition 2.6.6. $C_{L}(H)$ contains the semisimple and nilpotent parts of its elements.

Proposition 2.6.7. All semisimple elements of $C_{L}(H)$ lie in $H$.
Proposition 2.6.8. For all $\alpha, \beta \in H^{\star}$ we have that $\left[L_{\alpha}, L_{\beta}\right] \subseteq L_{\alpha+\beta}$.
Proposition 2.6.9. If $x \in L_{\alpha}$ such that $\alpha \neq 0$, then ad $x$ is nilpotent. In particular each $L_{\alpha}$ is nilpotent.

Proof. By the previous proposition we have that $\left[L_{\alpha}, L_{\alpha}\right] \subseteq L_{2 \alpha}$. Then for $x, y \in L_{\alpha}$ we have that $(a d h)([x, y])=[[h, x], y]+[x,[h, y]]=\alpha(h)[x, y]+\alpha(h)[x, y]=(a d h)([x, y])+$ $(a d h)([x, y])$ which implies that $(a d h)([x, y])=\{0\}$ and thus $L_{\alpha}$ must be nilpotent and therefore for every $x \in L_{\alpha}$ we have that $a d x$ is nilpotent by Engel's Theorem.

Proposition 2.6.10. If $\alpha, \beta \in H^{\star}$ and $\alpha+\beta \neq 0$ then $L_{\alpha}$ is orthogonal to $L_{\beta}$ relative to the Killing form $\kappa$ of $L$.

Proposition 2.6.11. The restriction of the Killing form to $C_{L}(H)\left(=L_{0}\right)$ is nondegenerate.

Proposition 2.6.12. The restriction of the Killing form $\kappa$ to $H$ is nondegenerate.

Proposition 2.6.13. $C_{L}(H)$ is nilpotent.
Proposition 2.6.14. If $x, y$ are commuting endomorphisms of a finite dimensional vector space such that $y$ is nilpotent, then $x y$ is nilpotent and in particular $\operatorname{tr}(x y)=0$.

Proposition 2.6.15. $C_{L}(H)$ is abelian.
With all our propositions in hand we finish off the proof as done by Humphreys.

Proof of theorem 2.6.5. [8] We want to show that $C_{L}(H)=H$. Suppose contrarily that $C_{L}(H)$ contains a nonzero nilpotent element, $x$, that is not in $H$. (This element must be nilpotent by propositions (2.6.6) and (2.6.7)). But then according to proposition (2.6.15) and (2.6.14) we have that $\kappa(x, y)=\operatorname{tr}(\operatorname{adx} \circ \operatorname{ady})=0$ for all $y \in C_{L}(H)$ which contradicts proposition (2.6.11). Thus no such $x$ exists and equality is proved.

Traditionally, a maximum toral subalgebra wasn't used to create the root space decomposition. Instead Cartan used a different subalgebra, later called the Cartan subalgebra, in order to create the root space decomposition.

Definition 2.6.16 (Cartan Subalgebra). A Cartan subalgebra of a Lie algebra $L$ is a self-normalising nilpotent subalgebra of $L$.

A Cartan subalgebra is usually established by finding the Fitting decomposition of a Lie algebra $L$. The Fitting decomposition is defined as $\left\{x \in L \mid(\operatorname{ad} h-\alpha I)^{m}(x)=\right.$ 0 for some $m \in \mathbb{N}\}$ where $I$ is the identity matrix and $a d h$ is a matrix representation. The Cartan subalgebra would be $L_{0}$ in the Fitting decomposition.

Theorem 2.6.17. Let $L$ be a semisismple Lie algebra. The Cartan subalgebra of $L$ is the same as the maximal toral subalgebra of $L$.

Proof. We already saw that a maximal toral subalgebra $H$ is nilpotent (proposition (2.6.13)) thus we only need to show that $H$ is self-normalising. To do this we must show that $H=N_{L}(H)=\{n \in L \mid[n, h] \in H \quad \forall h \in H\}$. It is easy to see that $H \subseteq N_{L}(H)$ thus we need to show the converse. Let $x \in N_{L}(H)$. We have that $[x, h] \in H$ for all $h \in H$. But as $H$ is abelian, $[x, h]=0$ and therefore $[x, h] \in H$. Thus $H$ is self-normalising.

Next we show that a Cartan subalgebra $C$ is a maximal toral subalgebra. We must show that $C$ consists of semisimple parts and is maximal with these qualities. First let's show that a Cartan subalgebra is maximal. (Proof of maximality as described in [15]) Suppose $C^{\prime}$ is another Cartan subalgebra such that $C^{\prime} \supseteq C$. Since $C^{\prime}$ is nilpotent there is some integer $m$ such that $\left(C^{\prime}\right)^{m}=0$. In particular for any $m$ elements in $C^{\prime}$ we must have

$$
\left[c_{m}^{\prime},\left[\ldots\left[c_{3}^{\prime},\left[c_{2}^{\prime}, c_{1}^{\prime}\right]\right] \ldots\right]\right]=0
$$

We can take any $c \in C$ and $c^{\prime} \in C^{\prime}$ such that

$$
\underbrace{[c,[\ldots,[c,[c,}_{m-1}, c^{\prime}]] \ldots]]=0 .
$$

Thus we see that $(a d c)^{m-1}\left(c^{\prime}\right)=0$. But this means that $c^{\prime}$ is in $L_{0}=C$ in the Fitting decomposition of $L$. Thus we have $C^{\prime} \subseteq C$ therefore $C=C^{\prime}$ and $C$ is maximal.

Lastly we must show that every element of $C$ is semisimple. We prove this by the method found in [12]. Let $x=x_{s}+x_{n}$ be the Jordan-Chavelley decomposition of $x$. Now if $y \in C$ then we have that $[x, y]=0$ and thus we also have that $\left[x_{s}, y\right]=\left[x_{n}, y\right]=$ 0 . We therefore have that $x_{s}, x_{n} \in C_{L}(C)=C$. But since $x_{n}$ and $y$ commute and $a d x_{n}$ is nilpotent, then $a d y \circ a d x_{n}$ is also nilpotent and thus has trace 0 . Therefore $x_{n}$ is orthogonal to every element of $C$. And since $x_{n} \in C$ we must have that $x_{n}=0$. Therefore we see that $x=x_{s}$ showing that $x$ must be semisimple.

Therefore we allow $H$ to represent both the maximal toral subalgebra and the Cartan subalgebra. We might wonder whether or not a Cartan decomposition is even possible. From proposition (2.3.4) we see that every Lie algebra has a semisimple subalgebra allowing us to construct a maximal one, giving us the decomposition we need.

### 2.7 Weight Spaces ${ }^{[8]}$

In order to classify all semisimple Lie algebras we first take a detour to define weights. In this section we let $\mathfrak{s}$ be the Lie algebra $\mathfrak{s l}(2, F)$ seen in section 2.4 where $F$ is a finite dimensional algebraically closed field with characteristic $0 . \mathfrak{s}$ has the standard basis $e=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right), h=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right), f=\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right)$ and such that $[e, h]=-2 e,[f, h]=2 f,[e, f]=h$.

Definition 2.7.1 (Weight Space). We let $V$ be an arbitrary $\mathfrak{s}$-module. Noting that $h$ is semisimple we can define $V_{\lambda}:=\{v \in V \mid h . v=\lambda v\}$ to be the eigenspace of $\lambda \in F$. We say that $\lambda$ is a weight of $h$ in $V$ if $V_{\lambda} \neq 0$ and we call $V_{\lambda}$ the weight space.

As $\mathfrak{s}$ is only three dimensional, it is easy to deduce the following lemma (proof can be found in [8]).

Lemma 2.7.2. If $v \in V_{\lambda}$ then $e . v \in V_{\lambda+2}$ and $f . v \in V_{\lambda-2}$.

Any nonzero vector in $V_{\lambda}$ annihilated by $e$ is called a maximal vector of weight $\lambda$. We state the following without proof. Proofs can be found in [8].

Lemma 2.7.3. Let $V$ be an irreducible $\mathfrak{s}$-module and let $v_{0} \in V_{\lambda}$ be a maximal vector. Let $v_{i-1}=0$ and $v_{i}=\frac{1}{i!} f^{i} \cdot v_{0}(i \geq 0)$. Then
(i) $h \cdot v_{i}=(\lambda-2 i) v_{i}$
(ii) f. $v_{i}=(i+1) v_{i+1}$
(iii) $e . v_{i}=(\lambda-i+1) v_{i-1}(i \geq 0)$

Theorem 2.7.4. Let $V$ be an irreducible $\mathfrak{s}$-module.
(i) Relative to $h, V$ is the direct sum of weight spaces $V_{\mu}$, where $\mu=m, m-$ $2, \ldots,-(m-2),-m$ such that $m+1=\operatorname{dim} V$ and $\operatorname{dim} V_{\mu}=1$ for each $\mu$.
(ii) $V$ has a unique maximal vector up to nonzero scalar multiples such that its weight is $m$. This weight is known as the highest weight of $V$.
(iii) There exists at most one irreducible $\mathfrak{s}$-module of each possible dimension $m+1>$ 0 up to isomorphism.

Corollary 2.7.5. Let $V$ be a finite dimensional $\mathfrak{s}$-module. Then the eigenvalues of $h$ on $V$ are all integers and each occurs along with its negative an equal number of times. Additionally, in any decomposition of $V$ into a direct sum of irreducible submodules, the number of summands is exactly $\operatorname{dim} V_{0}+\operatorname{dim} V_{1}$.

### 2.8 A new Euclidean space ${ }^{[8]}$

From section 2.6 we saw that a semisimple Lie algebra has a Cartan decomposition

$$
L=H \oplus \bigoplus_{\alpha \in \Phi} L_{\alpha}
$$

where $H=L_{0}$ is the maximal toral subalgebra. From proposition 2.6 .12 we see that it is possible to identify $H$ with $H^{\star}$. We see this by associating for every $\phi \in H^{\star}$ a unique element $t_{\phi} \in H$ such that $\kappa\left(t_{\phi}, h\right)=\phi(h)$ for all $h \in H$. We can thus associate our set $\Phi$ from our Cartan decomposition with the set $\left\{t_{\alpha} \mid \alpha \in \Phi\right\} \subseteq H$.

From here it is fairly easy to prove the following proposition. A proof can be found in [8].

Proposition 2.8.1. (i) $\Phi$ spans $H^{\star}$
(ii) $\alpha \in \Phi \Rightarrow-\alpha \in \Phi$
(iii) If we let $\alpha \in \Phi, e \in L_{\alpha}, f \in L_{-\alpha}$. Then $[e, f]=\kappa(e, f) t_{\alpha}$
(iv) If $\alpha \in \Phi$ then $\left[L_{\alpha}, L_{-\alpha}\right]$ is one dimensional with basis $\left\{t_{\alpha}\right\}$
(v) $\alpha\left(t_{\alpha}\right)=\kappa\left(t_{\alpha}, t_{\alpha}\right) \neq 0, \forall \alpha \in \Phi$
(vi) If $\alpha \in \Phi$ and $e_{\alpha} \in L_{\alpha}$ is nonzero, then there exists $f_{\alpha} \in L_{-\alpha}$ and $h_{\alpha}=\left[e_{\alpha}, f_{\alpha}\right]$ such that $e_{\alpha}, h_{\alpha}, f_{\alpha}$ span a three dimensional simple subalgebra $\mathfrak{s}_{\alpha}$ of $L$ such that $\mathfrak{s}_{\alpha} \cong \mathfrak{s l}(2, F)$ via $e_{\alpha} \mapsto\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right), h_{\alpha} \mapsto\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$, and $f_{\alpha} \mapsto\left(\begin{array}{cc}0 & 0 \\ 1 & 0\end{array}\right)$
(vii) $h_{\alpha}=\frac{2 t_{\alpha}}{\kappa\left(t_{\alpha}, t_{\alpha}\right)} ; h_{\alpha}=-h_{-\alpha}$

From here we can deduce the following (proof outlined in [8]).
Proposition 2.8.2. (i) $\alpha \in \Phi$ implies that $\operatorname{dim} L_{\alpha}=1$. In particular if we let $H_{\alpha}=\left[L_{\alpha}, L_{-\alpha}\right]$ we see that $\mathfrak{s}_{\alpha}=L_{\alpha}+L_{-\alpha}+H_{\alpha}$ and for any given nonzero $e_{\alpha} \in L_{\alpha}$ there is a unique $f_{\alpha} \in L_{-\alpha}$ such that $h_{\alpha}=\left[e_{\alpha}, f_{\alpha}\right]$.
(ii) If $\alpha \in \Phi$, then $k \alpha \in \Phi$ if $k= \pm 1$.
(iii) If $\alpha, \beta \in \Phi$, then $\beta\left(h_{\alpha}\right) \in \mathbb{Z}$ and $\beta-\beta\left(h_{\alpha}\right) \alpha \in \Phi$.
(iv) If $\alpha, \beta, \alpha+\beta \in \Phi$, then $\left[L_{\alpha}, L_{\beta}\right]=L_{\alpha+\beta}$.
(v) If $\alpha, \beta \in \Phi$ such that $\beta \neq \pm \alpha$ and we let $r$ and $q$ be the largest integers such that $\beta-r \alpha$ and $\beta+q \alpha$ are roots respectively, then all $\beta+i \alpha \in \Phi$ and $\beta\left(h_{\alpha}\right)=r-q$ where $-r \leq i \leq q$.
(vi) $L$ is generated by the root spaces $L_{\alpha}$.

From here we can start our tie in with reflection groups. First we need to set up a Euclidean space to work in.

Let $L$ be a semisimple Lie algebra over an algebraically closed field $F$ of characteristic 0 with $H$ its maximal toral subalgebra and $\Phi \subseteq H^{\star}$ be the roots of $L$ as described in a Cartan decomposition. We know that the Killing form restricted to $H$ is nondegenerate by proposition 2.6.12. We then transfer the Killing form over to $H^{\star}$, as $H \cong H^{\star}$, by letting $(\phi, \psi)=\kappa\left(t_{\phi}, t_{\psi}\right)$ for all $\phi, \psi \in H^{\star}$. As $\Phi$ spans all of $H^{\star}$ we know that we can choose a basis of $H^{\star}$ consisting solely of roots. Let this basis be $\left\{\alpha_{1}, \alpha_{2}, \ldots \alpha_{n}\right\}$. Therefore, for any $\beta \in \Phi$ we see there is a unique way of expanding $\beta$ such that $\beta=\sum_{i=1}^{n} c_{i} \alpha_{i}$ where $c_{i} \in F$.

Claim: $\quad c_{i} \in \mathbb{Q}$ and are unique for a particular $\beta$.
Proof. (Outlined in [8]) It is easy to see

$$
\left(\beta, \alpha_{j}\right)=\sum_{i=1}^{n} c_{i}\left(\alpha_{i}, \alpha_{j}\right) \quad \forall j \in\{1, \ldots, n\} .
$$

Thus multiplying both sides by a term $2 /\left(\alpha_{j}, \alpha_{j}\right)$ we get

$$
\frac{2\left(\beta, \alpha_{j}\right)}{\left(\alpha_{j}, \alpha_{j}\right)}=\sum_{i=1}^{n} \frac{2\left(\alpha_{i}, \alpha_{j}\right)}{\left(\alpha_{j}, \alpha_{j}\right)} c_{i} \quad \forall j \in\{1, \ldots, n\}
$$

We thus have $n$ equations with $n$ unknowns (the $c_{i}$ ). As each of these are in $\Phi$ we know that the $c_{i}$ must be rational and thus in $\mathbb{Q}$.

Now all that is left is to show uniqueness. As $H^{\star}$ has basis $\left\{\alpha_{1}, \alpha_{2}, \ldots \alpha_{n}\right\}$ and since the form is nondegenerate we know that the matrix with entries ( $\alpha_{a}, \alpha_{b}$ ) ( $a$-th row, $b$-th column) must be nonsingular. From here we know that the coefficient matrix of this matrix must also be nonsingular. Thus our $c_{i}$ must be unique over $\mathbb{Q}$.

We now let $E_{\mathbb{Q}}$ be the $\mathbb{Q}$-subspace of $H^{\star}$ spanned by all the roots in $\Phi$. Thus we see that $\operatorname{dim}_{F} H^{\star}=n$. To transition to an Euclidean space let's look at the inner product more closely. We recall that for any $\phi, \psi \in H^{\star}$ we have $(\phi, \psi)=$ $\kappa\left(t_{\phi}, t_{\psi}\right)=\sum_{\alpha \in \Phi} \alpha\left(t_{\phi}\right) \alpha\left(t_{\psi}\right)=\sum_{\alpha \in \Phi}(\alpha, \phi)(\alpha, \psi)$. Now let $\beta \in \Phi \subseteq H^{\star}$ be another root. We notice that $(\beta, \beta)=\sum(\alpha, \beta)^{2}$. And thus dividing both sides by $(\beta, \beta)^{2}$ we get $1 /(\beta, \beta)=\sum(\alpha, \beta)^{2} /(\beta, \beta)^{2}$. From our previous proposition we know that $2(\alpha, \beta)^{2} /(\beta, \beta)^{2} \in \mathbb{Z}$ and therefore we see that $(\beta, \beta) \in \mathbb{Q}$. Thus every inner product of vectors in $E_{\mathbb{Q}}$ are rational and thus we have a nondegenerate form on $E_{\mathbb{Q}}$. Now taking any arbitrary $\nu \in E_{\mathbb{Q}}$ we see that $(\nu, \nu)=\sum(\alpha, \nu)^{2}$. Therefore we see that the form must be positive definite on $E_{\mathbb{Q}}$.

Finally we extend the base field from $\mathbb{Q}$ to $\mathbb{R}$. We let $E$ be the real vector space obtained by this extensions and we see that $E=\mathbb{R} \otimes E_{\mathbb{Q}}$. The form above can be seen to be a positive definite symmetric bilinear form that can be extended to $E$ as well. Thus we see that $E$ can be defined as a Euclidean space such that $\Phi$ contains a basis of $E$ and $\operatorname{dim}_{\mathbb{R}} E=n$. This dimension of $E$ is known as the rank of the root system $\Phi$. The previous is summarised in the theorem below.

Theorem 2.8.3. Letting $L, H, \Phi, E$ be constructed as described above
(i) $\Phi$ spans $E$ and $0 \notin \Phi$.
(ii) If $\alpha \in \Phi$ then $k \alpha \in \Phi$ implies that $k= \pm 1$.
(iii) If $\alpha, \beta \in \Phi$ then $\beta-\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha \in \Phi$.
(iv) If $\alpha, \beta \in \Phi$ then $\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

Corollary 2.8.4. $\operatorname{dim} L=\operatorname{rank}(L)+|\Phi|$
Proof. This falls straight away as the rank of $L$ is just the dimension of its Cartan subalgebra plus the dimension of each root space, each of which have dimension one, and which we have $|\Phi|$ of.

By looking at theorem 2.8.3 we see startling similarities with reflections. We recall that $s_{\alpha}(\nu)=\nu-\frac{2(\nu, \alpha)}{(\alpha, \alpha)} \alpha$ and that the only scalar multiples of $\alpha$ that produced reflections was 1 . We might therefore wonder if the root system of a semisimple Lie algebra can be seen to be a type of reflection group. This turns out to be the case.

## Chapter 3

## Root Systems

### 3.1 Definitions ${ }^{[8],[9]}$

We saw that a reflection group is a group generated by reflections $s_{\alpha}$. We now move our focus onto the $\alpha \in E$, where $E$ is the real Euclidean space as defined in 2.8.

Definition 3.1.1 (Root System). The finite set of all nonzero $\alpha \in E$, denoted as $\Phi$, which span $E$ is known as a root system if:
(R1) If $\alpha \in \Phi$ then $k \alpha \in \Phi$ implies $k= \pm 1$.
(R2) $s_{\alpha} \Phi=\Phi$ for all $\alpha \in \Phi$ (the reflection leaves $\Phi$ invariant).

Each $\alpha$ in a root system $\Phi$ is known as a root. We introduce a short hand

$$
\langle\beta, \alpha\rangle=\frac{2(\beta, \alpha)}{(\alpha, \alpha)}
$$

where $\langle\beta, \alpha\rangle$ is known as a Cartan integer when $\langle\beta, \alpha\rangle \in \mathbb{Z}$.

Definition 3.1.2. We say that a root system is crystallographic if it satisfies the above two properties for a root system and simultaneously satisfies:
(R3) For all $\alpha, \beta \in \Phi,\langle\beta, \alpha\rangle$ is a Cartan integer.

As we saw in theorem 2.8.3(iv), the root system for a semisimple Lie algebra is such that each $\langle\beta, \alpha\rangle \in \mathbb{Z}$. Therefore every root system of a semisimple Lie algebra is crystallographic.

Definition 3.1.3 (Weyl Group). Let $W$ be a subgroup of $G L(E)$ generated by reflections of roots in a root system $\Phi$. As $\Phi$ is finite and spans all of $E$, and since $W$ permutes $\Phi$ by invariance we see that $W$ is a subgroup of the symmetric group on $\Phi$. Thus $W$ is essential and is finite. $W$ is known as the Weyl group.

It is possible to define the dual of a root system. We define a dual root as

$$
\alpha^{\vee}=\frac{2 \alpha}{(\alpha, \alpha)}
$$

and define the dual of a root system $\Phi$ as

$$
\Phi^{\vee}=\left\{\alpha^{\vee} \mid \alpha \in \Phi\right\}
$$

This $\Phi^{\vee}$ is itself a root system and is generally isomorphic to $\Phi$ except in a few cases.
Using roots we can also construct a lattice. We define the root lattice of $\Phi$ in $E$ to be the $\mathbb{Z}$-span of $\Phi$ denoted $L(\Phi)$. Similarly we can define a dual root lattice $L\left(\Phi^{\vee}\right)$. We can then define the weight lattice and the dual weight lattice as $\hat{L}(\Phi):=$ $\left\{\nu \in E \mid\left(\nu, \alpha^{\vee}\right) \in \mathbb{Z}\right.$ for all $\left.\alpha \in \Phi\right\}$ and $\hat{L}\left(\Phi^{\vee}\right):=\{\nu \in E \mid(\nu, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi\}$ respectively. It can be seen that any crystallographic root system preserves a lattice.

### 3.2 Positive and Simple Roots ${ }^{[9],[15]}$

With our definition of root system we see that the only two multiples of a root $\alpha$ in a root system $\Phi$ are $\alpha$ and $-\alpha$. We therefore try and break $\Phi$ into two subsets, one 'positive' and one 'negative'. To do this we must first define a total ordering on our vector space.

Definition 3.2.1. A strict total ordering in a set $X$ is a relation, denoted $<$, such that
(i) $<$ is transitive.
(ii) For every $x_{1}, x_{2} \in X$, either $x_{1}<x_{2}, x_{1}=x_{2}$, or $x_{1}>x_{2}$.

We say that a set has a partial ordering if we take away the restriction that every $x_{1}, x_{2}$ have a relation. In other words there might exist some $x_{1}$ and $x_{2}$ such that they neither satisfy $x_{1}<x_{2}, x_{1}=x_{2}$, nor $x_{1}>x_{2}$. With a strict total ordering on a vector space $V$ we can define an element $\nu \in V$ as positive if $0<\nu$.

From this definition we can create a subset $\Pi$ of $\Phi$ which is the set of all positive roots relative to some strict total ordering. We call $\Pi$ a positive system and the associated system, $-\Pi$, a negative system. As every root comes in pairs we see that for each root either $\alpha$ or $-\alpha$ will be in $\Pi$ (and the other will be in $-\Pi$ ) and thus $\Pi$ and $-\Pi$ are disjoint and therefore $\Phi=\Pi \oplus-\Pi$.

From here we might wonder if we can use the elements of $\Pi$ to form a basis for $\Phi$. Although not trivial, this can be shown to be the case.

Definition 3.2.2 (Simple System). A simple system is a subset $\Delta=\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}$ of a root system $\Phi$ such that
(i) The elements of $\Delta$ are linearly independent.
(ii) For any $\beta \in \Phi, \beta$ can be written as a linear combination of elements from $\Delta$ with coefficients all of the same sign

$$
\beta= \pm\left(a_{1} \alpha_{1}+a_{2} \alpha_{2}+\ldots+a_{n} \alpha_{n}\right) \quad a_{i} \in \mathbb{Z}_{\geq 0}, \alpha_{i} \in \Delta, i \in\{1,2, \ldots, n\}
$$

As the existence of a simple system is non-trivial we must show they exist.
Theorem 3.2.3. Let $\Delta$ be a simple system and $\Pi$ a positive system in root system $\Phi$.
(i) For any $\Delta$ there is a unique $\Pi$ which contains $\Delta$.
(ii) Every $\Pi$ contains a unique $\Delta$.

Proof. (Expanded and restructured from [9])
(i) Let's first suppose that $\Delta \subseteq \Pi$ for some $\Pi$. Then all roots which are nonnegative linear combinations of $\Delta$ are in $\Pi$. The negatives of these roots are not in $\Pi$ and by disjointness $\Pi$ is unique with these roots. For existence we first extend our linearly independent set $\Delta$ to an ordered basis for $E$. We know this ordered basis is $\Phi$. Thus we can just take the positive elements from $\Phi$ and call them $\Pi$ where positive is defined with a lexicographical ordering. Thus we see that $\Delta \subseteq \Pi$.
(ii) As before, we first prove uniqueness. Let's suppose that a positive system $\Pi$ does in fact have a simple system $\Delta$ contained in it. As $\alpha \in \Delta$ can be viewed as a root such that no linear combination with strictly positive coefficients exists,
we can see that $\Delta$ can be defined uniquely and is thus the unique simple system associated with $\Pi$.

Finally, we show the existence of a simple system. To do this we first take as small a subset $\Delta \subseteq \Pi$ as possible such that each root in $\Pi$ is a nonnegative linear combination of the elements of $\Delta$. This subset must exist (and could even be the whole of $\Pi$ ). Therefore our only problem is to show that the elements of $\Delta$ are linearly independent. To do this we want to show that the following is true

$$
\begin{equation*}
(\beta, \alpha) \leq 0 \quad \forall \alpha \neq \beta, \alpha, \beta \in \Delta \tag{3.1}
\end{equation*}
$$

Contrarily, suppose that (3.1) is false and thus breaks down for some $\alpha, \beta \in \Delta$. Our reflection formula gives us $s_{\alpha}(\beta)=\beta-\langle\beta, \alpha\rangle \alpha$. As $(\beta, \alpha)>0$ this implies that $\langle\beta, \alpha\rangle>0$. By definition of a root system $s_{\alpha}(\beta) \in \Phi$ and therefore $s_{\alpha}(\beta)$ is in either $\Pi$ or $-\Pi$. Thus we have 2 cases:
(a) Let's assume that $s_{\alpha}(\beta) \in \Pi$ (and is therefore positive). As any element in $\Phi$ is a linear combination of the elements of $\Delta$ we see that $s_{\alpha}(\beta)$ can be defined as $\sum_{\gamma \in \Delta, c_{\gamma} \geq 0} c_{\gamma} \gamma$. Thus $s_{\alpha}(\beta)=\beta-\langle\beta, \alpha\rangle \alpha=c_{\beta} \beta+\sum_{\gamma \neq \beta} c_{\gamma} \gamma$. If $c_{\beta}<1$ we see that $\left(1-c_{\beta}\right) \beta=\langle\beta, \alpha\rangle \alpha+\sum_{\gamma \neq \beta} c_{\gamma} \gamma$ and thus $\beta$ is a nonnegative linear combination of $\Delta \backslash\{\beta\}$ which contradicts minimality of $\Delta$. Therefore $c_{\beta} \geq 1$. Thus we see that $0=\left(c_{\beta}-1\right) \beta+\langle\beta, \alpha\rangle \alpha+\sum_{\gamma \neq \beta} c_{\gamma} \gamma$. But as all of these roots are positive and at least one coefficient is nonzero this implies the right hand side cannot equal 0 therefore we see that $s_{\alpha}(\beta)$ cannot be positive.
(b) Let's assume that $s_{\alpha}(\beta) \in-\Pi$ (and is therefore negative). Thus we know that $-s_{\alpha}(\beta) \in \Pi$. As before we see that $-s_{\alpha}(\beta)=-\beta+\langle\beta, \alpha\rangle \alpha=c_{\alpha} \alpha+$ $\sum_{\gamma \neq \alpha} c_{\gamma} \gamma$. If $c_{\alpha}<\langle\beta, \alpha\rangle$ we see that $\left(\langle\beta, \alpha\rangle-c_{\alpha}\right) \alpha=\alpha+\sum_{\gamma \neq \alpha} c_{\gamma} \gamma$ and thus $\alpha$ is a nonnegative linear combination of $\Delta \backslash\{\alpha\}$ which contradicts minimality of $\Delta$. Therefore $c_{\alpha} \geq\langle\beta, \alpha\rangle$. Thus we see that $0=\left(c_{\alpha}-\right.$ $\langle\beta, \alpha\rangle) \alpha+\beta+\sum_{\gamma \neq \alpha} c_{\gamma} \gamma$. But as all of these roots are positive and at least one coefficient is nonzero this implies the right hand side cannot equal 0 therefore we see that $s_{\alpha}(\beta)$ cannot be negative either.

This contradiction implies that (3.1) is true.

Now let's look at what happens if $\Delta$ is not linearly independent. Then $\sum_{\alpha \in \Delta} a_{\alpha} \alpha=$ 0 with not all $a_{\alpha}=0$ by definition of linear dependence. Rewrite this by taking sums over disjoint sets of $\Delta$ with strictly positive coefficients. We get $\sum b_{\beta} \beta=\sum c_{\gamma} \gamma=$ $\sigma>0$. But by (3.1) we see that

$$
0 \leq(\sigma, \sigma)=\left(\sum b_{\beta} \beta, \sum c_{\gamma} \gamma\right) \leq 0
$$

and thus $\sigma=0$ which is a contradiction. Therefore $\Delta$ is linearly independent and thus a simple system does exist.

Corollary 3.2.4. For all $\alpha, \beta \in \Delta \subseteq \Pi$, if $\alpha \neq \beta$ then

$$
(\beta, \alpha) \leq 0
$$

As simple systems do exist this motivates us to define each $\alpha \in \Delta$ as a simple root and its associated reflection $s_{\alpha}$ as a simple reflection.

### 3.3 Coxeter Groups ${ }^{[3],[9]}$

We now discuss Coxeter groups as they can be helpful in deducing Weyl groups.
Definition 3.3.1. Let $\mathscr{S}$ be a set of generators for a multiplicative group $\mathscr{W}$ such that $1 \notin \mathscr{S}, \mathscr{W}$ is finitely generated by $\mathscr{S}$ and every element of $\mathscr{S}$ has order 2 . We also let $m\left(s, s^{\prime}\right)$ denote the order of $s s^{\prime}$ for all $s, s^{\prime} \in \mathscr{S}$ and define $\mathscr{I}$ as the set of pairs such that $m\left(s, s^{\prime}\right)$ is finite. We define $(\mathscr{W}, \mathscr{S})$ to be Coxeter system if the generating set $\mathscr{S}$ and the relations $\left(s s^{\prime}\right)^{m\left(s, s^{\prime}\right)}=1$ for $\left(s, s^{\prime}\right) \in \mathscr{I}$ form a presentation of a group $\mathscr{W}$. We call $\mathscr{W}$ a Coxeter group.

Note that we can't simply refer to a Coxeter group as the group $\mathscr{W}$ without also referring to what reflections generate it. This is due to the fact that a single group might be generated by two different reflection groups and thus the Coxter system would be different and our theory would break down. A quick example can be found in [1] where we see the dihedral group of order 12 can be represented as

$$
\begin{aligned}
& D_{6} \cong\left\langle a, b \mid a^{2}=b^{2}=(a b)^{6}=1\right\rangle \\
& D_{6} \cong\left\langle x, y, z \mid x^{2}=y^{2}=z^{2}=(x y)^{3}=(x z)^{2}=(y z)^{2}=1\right\rangle
\end{aligned}
$$

The first Coxeter system has type $I_{2}(6)$ and the second one has type $A_{2} \times A_{1}$. Thus we need to ensure that our abuse of language doesn't get abused.

We can easily see that a reflection group is the same as a Coxeter group since each reflection has order 2 and its simple system of reflections generates the entire reflection group. As Weyl groups are essential reflection groups relative to a Euclidean space $E$ we see that Weyl groups are just a type of Coxeter groups. Note that we will use $\mathscr{W}$ to denote Coxeter groups and $W$ to denote Weyl groups.

As simple and positive systems don't rely on $W$ being essential, we can extend those concepts to Coxeter groups. Therefore given a root system $\Phi$ we can find a simple system $\Delta$ and from there construct a generating set $\mathscr{S}$ such that for all $\alpha \in \Delta$ we have $s_{\alpha} \in \mathscr{S}$. Therefore $\mathscr{S}$ is our set of reflections which generate our reflection group $\mathscr{W}$. For shorthand we will let $s_{i}=s_{\alpha_{i}}$ for $\alpha_{i} \in \Delta$.

Definition 3.3.2. For any $w \in \mathscr{W}$ we define the length of $w$ to be the smallest integer $q \geq 0$ such that $w$ is a product of a sequence $q$ elements of $\mathscr{S}$. The length is denoted $\ell(w)$ or $\ell_{\mathscr{S}}(w)$ and is such that $\ell(w)=q$ when $w=s_{1} \ldots s_{q}$ with sequence $\boldsymbol{s}=\left(s_{1}, \ldots, s_{q}\right)$ of elements of $\mathscr{S}$. This sequence of smallest $q$ is known as the reduced decomposition of $w$ with respect to $\mathscr{S}$. By convention we let $\ell(1)=0$.

Proposition 3.3.3. Let $w$ and $w^{\prime}$ be in $\mathscr{W}$. We see that
(i) $\ell\left(w w^{\prime}\right) \leq \ell(w)+\ell\left(w^{\prime}\right)$
(ii) $\ell\left(w^{-1}\right)=\ell(w)$
(iii) $\left|\ell(w)-\ell\left(w^{\prime}\right)\right| \leq \ell\left(w w^{\prime-1}\right)$

We let $\mathscr{T}$ be the set of conjugates in $\mathscr{W}$ of elements in $\mathscr{S}$. For any sequence $\boldsymbol{s}=\left(s_{1}, \ldots, s_{q}\right)$ of elements in $\mathscr{S}$ we denote by $\tau(\boldsymbol{s})$ the sequence $\left(t_{1}, \ldots, t_{q}\right)$ of elements of $\mathscr{T}$ where for $1 \leq j \leq q$ we have $t_{j}=\left(s_{1} \ldots s_{j-1}\right) s_{j}\left(s_{1} \ldots s_{j-1}\right)^{-1}$.

For any $t \in \mathscr{T}$ let $n(s, t)$ denote the number of integers $j$ such that $1 \leq j \leq q$ and $t_{j}=t$.

Lemma 3.3.4. If $t \in O(E)$ and $\alpha$ is a nonzero vector in $E$ the $t s_{\alpha} t^{-1}=s_{t \alpha}$.
Proof. [9] It's fairly easy to see that $t s_{\alpha} t^{-1}$ sends $t \alpha$ to its negative. Thus we only need to show that $t s_{\alpha} t^{-1}$ fixes pointwise the hyperplane of $\alpha$. Recall that $\eta$ is in the
hyperplane of $\alpha$ if and only if $t \eta$ is in the hyperplane of $t \alpha$. This is because $(\eta, \alpha)=$ $(t \eta, t \alpha)$. Thus we see that if $\eta$ is in the hyperplane of $\alpha$ then $\left(t s_{\alpha} t^{-1}\right)(t \eta)=t s_{\alpha} \eta=t \eta$. This gives us our lemma.

We also define $\pi(w):=\operatorname{Card}\left(\Pi \cap w^{-1}(-\Pi)\right)$ as the number of positive roots that get sent to its negative by $w$ and define $\delta(w):=(-1)^{\ell(w)}$ as the determinant of $w$. As $\delta$ is a homomorphism we see that $\delta\left(w w^{\prime}\right)=\delta(w) \delta\left(w^{\prime}\right)$ implies that $\ell\left(w w^{\prime}\right) \equiv$ $\ell(w)+\ell\left(w^{\prime}\right)(\bmod 2)$.

We state without proof the following lemma. A proof may be found in [9].
Lemma 3.3.5. Let $\Delta$ be a simple system such that $\alpha \in \Delta$. Then $s_{\alpha}(\Pi \backslash\{\alpha\})=\Pi \backslash\{\alpha\}$.
Lemma 3.3.6. Let $s_{\alpha} \in \mathscr{S}$ for some $\alpha \in \Delta$ and $w \in \mathscr{W}$. Then the following are true
(i) If $w \alpha>0$ then $\pi\left(w s_{\alpha}\right)=\pi(w)+1$.
(ii) If $w \alpha<0$ then $\pi\left(w s_{\alpha}\right)=\pi(w)-1$.
(iii) If $w^{-1} \alpha>0$ then $\pi\left(s_{\alpha} w\right)=\pi(w)+1$.
(iv) If $w^{-1} \alpha<0$ then $\pi\left(s_{\alpha} w\right)=\pi(w)-1$.

Proof. (Outlined in [9]) For ease we will let $\Pi(w):=\Pi \cap w^{-1}(-\Pi)$ therefore giving us that $\pi(w)=\operatorname{Card}(\Pi(w))$.
(i) If $w \alpha>0$ we see that $w$ doesn't reflect $\alpha$ and thus $w s_{\alpha}$ sends $\alpha$ to its negative. Thus we can see that $\Pi\left(w s_{\alpha}\right)$ is the disjoint union of $s_{\alpha} \Pi(w)$ and $\{\alpha\}$. Also as $w$ doesn't reflect $\alpha$ we see that $\alpha \notin \Pi(w)$ and therefore $\operatorname{Card}\left(s_{\alpha} \Pi(w)\right)=$ $\operatorname{Card}(\Pi(w))$. Therefore we see that $\pi\left(w s_{\alpha}\right)=\pi(w)+1$.
(ii) If $w \alpha<0$ then $w$ reflects $\alpha$ and thus $w s_{\alpha}$ makes it so that $\alpha$ is not reflected. Thus $\Pi\left(w s_{\alpha}\right)=\Pi(w) \backslash\{\alpha\}$ which leads to $\pi\left(w s_{\alpha}\right)=\pi(w)-1$.
(iii) Notice first off that

$$
\begin{aligned}
\Pi\left(w^{-1} s_{\alpha}\right) & =\Pi \cap s_{\alpha} w(-\Pi) \\
& =s_{\alpha}\left(s_{\alpha} \Pi \cap w(-\Pi)\right) \\
& =s_{\alpha} w^{-1}\left(w^{-1} s_{\alpha} \Pi \cap(-\Pi)\right) \\
& =-s_{\alpha} w^{-1}\left(\Pi \cap w^{-1} s_{\alpha}(-\Pi)\right) \\
& =-s_{\alpha} w^{-1} \Pi\left(s_{\alpha} w\right)
\end{aligned}
$$

and therefore $\pi\left(w^{-1} s_{\alpha}\right)=\pi\left(s_{\alpha} w\right)$. Now since $w^{-1} \alpha>0$ we see that $w \alpha>0$ as well and therefore $\pi\left(w^{-1} s_{\alpha}\right)=\pi\left(s_{\alpha} w\right)=\pi(w)+1$.
(iv) Similarly we see that since $w^{-1} \alpha<0$ then $w \alpha<0$. Thus $\pi\left(w^{-1} s_{\alpha}\right)=\pi\left(s_{\alpha} w\right)=$ $\pi(w)-1$.

Theorem 3.3.7. Let $\Delta$ be a simple system and $w=s_{1} \ldots s_{r}$ be any expression of $w \in \mathscr{W}$ where $s_{i} \in \mathscr{S}$. If $\pi(w)<r$ then there are indices $1 \leq i<j \leq r$ such that
(i) $\alpha_{i}=\left(s_{i+1} \ldots s_{j-1}\right) \alpha_{j} \quad \alpha_{i} \in \Delta$.
(ii) $s_{i+1} s_{i+2} \ldots s_{j}=s_{i} s_{i+1} \ldots \ldots s_{j-1}$
(iii) $w=s_{1} \ldots s_{i-1} s_{i+1} \ldots s_{j-1} s_{j+1} \ldots s_{r}$

Proof. (Outlined in [9])
(i) Since we have $\pi(w)<r$ we know that there must be some $j \leq r$ such that $\left(s_{1} \ldots s_{j-1}\right) \alpha_{j}<0$ by lemma 3.3.6. Now as $\alpha_{j} \in \Delta$ we know $\alpha_{j}>0$ and therefore there is an index $i<j$ such that $s_{i}\left(s_{i+1} \ldots s_{j-1}\right) \alpha_{j}<0$ and $\left(s_{i+1} \ldots s_{j-1}\right) \alpha_{j}>0$. Now lemma 3.3.5 applied to the simple reflection $s_{i}$ tells us that $\left(s_{i+1} \ldots s_{j-1}\right) \alpha_{j}$ is made negative by $\alpha_{i}$.
(ii) Now let $\alpha=\alpha_{j}, w^{\prime}=s_{i+1} \ldots s_{j-1}$ such that $w^{\prime} \alpha=\alpha_{i}$ (as seen in the previous part). We know that by lemma 3.3.4 $w^{\prime} s_{\alpha} w^{\prime^{-1}}=s_{w^{\prime} \alpha}=s_{i}$ and therefore $\left(s_{i+1} \ldots s_{j-1}\right) s_{j}\left(s_{j-1} \ldots s_{i+1}\right)=s_{i}$. Multiply both sides by $s_{i+1} \ldots s_{j-1}$ on the right to get the desired form.
(iii) If we take $w=s_{1} \ldots s_{r}$ and by the previous part we replace $s_{i+1} \ldots s_{j}$ with $s_{i} \ldots s_{j-1}$ we get

$$
\begin{aligned}
w & =s_{1} \ldots s_{r} \\
& =s_{1} \ldots s_{i-1} s_{i} s_{i} \ldots s_{j-1} s_{j+1} \ldots s_{r} \\
& =s_{1} \ldots s_{i-1} s_{i+1} \ldots s_{j-1} s_{j+1} \ldots s_{r}
\end{aligned}
$$

thus completing our proof.

Part iii of theorem 3.3.7 is called the deletion condition as it allows us to delete any two simple reflections in order to get a reduced form (if such a deletion can be made). This allows us to state the following.

Corollary 3.3.8. If $w \in \mathscr{W}$ then $\pi(w)=\ell(w)$.
Proof. (Pieced from [9]) Let $w=s_{1} \ldots s_{r}$.
We first show that $\pi(w) \leq \ell(w)$. We do this by induction on $i$ where $w_{i}:=s_{1} \ldots s_{i}$. $w_{1}=s_{1}$ and thus $\pi\left(w_{1}\right)=\ell\left(w_{1}\right)=1$. Now let $\pi\left(w_{i-1}\right) \leq \ell\left(w_{i-1}\right)$ for some $i$. Now $w_{i}=w_{i-1} s_{i}$ and therefore by lemma 3.3.6 $\pi\left(w_{i}\right)=\pi\left(w_{i-1}\right) \pm 1$ while $\ell\left(w_{i}\right)=\ell\left(w_{i-1}\right)+1$. Thus we see $\pi\left(w_{i}\right) \leq \ell\left(w_{i}\right)$.

If $\pi(w)<\ell(w)=r$ then we see by the deletion condition that we can remove two simple reflections from $w$ giving us $\ell(w)=r-2$ which is a contradiction. Therefore $\pi(w) \geq \ell(w)$. Thus $\pi(w)=\ell(w)$.

Finally we state what is known as the exchange condition.
Theorem 3.3.9 (Exchange condition). Let $w=s_{1} \ldots s_{r}$ such that each $s_{i}$ is a simple reflection. If $\ell(w s)<\ell(w)$ for some simple reflections $s$ then there exists an index $i$ such that ws $=s_{1} \ldots s_{i-1} s_{i+1} \ldots s_{r}$.

In particular, $w$ has a reduced expression ending in $s$ if and only if $\ell(w s)<\ell(w)$. Proof. [9] From theorem 3.3.7 we see that $\ell(w s)<\ell(w)$ implies that $w \alpha<0$ and thus $w s=s_{1} \ldots s_{r} s$. From the deletion condition we can let $j=r+1$ which then would give us $w s=s_{1} \ldots s_{i-1} s_{i+1} \ldots s_{r}$ giving us our proof.

Proposition 3.3.10. Let $w \in \mathscr{W}, s \in \mathscr{S}$, and $\boldsymbol{s}=\left(s_{1}, \ldots, s_{q}\right)$ be a reduced decomposition of $w$. Then either
(i) $\ell(s w)=\ell(w)+1$ and $\left(s, s_{1}, \ldots, s_{q}\right)$ is a reduced decomposition of $s w$.
(ii) $\ell(s w)=\ell(w)-1$ and there exists a $j$ such that $1 \leq j \leq q$ where $\left(s_{1}, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{q}\right)$ is a reduced decomposition of $s w$ and $\left(s, s_{1}, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{q}\right)$ is a reduced decomposition of $w$.

Proof. [3] Let $w^{\prime}=w s$ as before. We see that $\left|\ell(w)-\ell\left(w^{\prime}\right)\right| \leq \ell(s)=1$ and thus we have two cases
(i) If we first look at $\ell\left(w^{\prime}\right)>\ell(w)$ then we see that $\ell\left(w^{\prime}\right)=r+1$ and $w^{\prime}=s_{1} \ldots s_{r} s$ and so $\left(s_{1}, \ldots, s_{r}, s\right)$ is a reduced decomposition of $w^{\prime}$.
(ii) We next look at $\ell\left(w^{\prime}\right) \leq \ell(w)$. By the exchange condition, there exists an integer $i$ such that $1 \leq i \leq r$ and $s_{i} \ldots s_{r} s=s_{i-1} \ldots s_{r-1} s_{r}$. Therefore we can rewrite $w$ as $w=s_{1} \ldots s_{i-1} s_{i+1} \ldots s_{r} s$ and as $s$ is of order two we see that $w s=s_{1} \ldots s_{i-1} s_{i+1} \ldots s_{r}$. As $r-1 \leq \ell\left(w^{\prime}\right) \leq r$ we see that it must be the case that $\ell\left(s^{\prime}\right)$ and that $\left(s_{1}, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{r}\right)$ is a reduced decomposition of $w^{\prime}$.

We state the following theorem which will not be rigorously proved.
Theorem 3.3.11. Fix a simple system $\Delta$ in $\Phi$. Then $\mathscr{W}$ is generated by the set $\mathscr{S}=\left\{s_{\alpha} \mid \alpha \in \Delta\right\}$ using only the relations $\left(s_{\alpha}, s_{\beta}\right)^{m(\alpha, \beta)}=1$ for all $\alpha, \beta \in \Delta$.
Proof. (Outline of proof from [9]) We want to show that each relation in $\mathscr{W}$ is a consequence of the given relations, or that $s_{\alpha_{1}} \ldots s_{\alpha_{r}}=1$ for some $\alpha_{i} \in \Delta$. As before we let $s_{i}=s_{\alpha_{i}}$ for $\alpha_{i} \in \Delta$. We know that $r$ must be even as the determinant of the reflections is -1 .

We prove the theorem by induction on $r$. The case of $r=2$ is trivial and leads to $\left(s_{1}\right)^{2}=1$ giving us our relations. Now let $r=2 q$ for some $q>1$. We consistently use the exchange condition to try and get a reduced form. Our relation changes as follows

Given relation where $r=2 q$ gives

$$
\begin{aligned}
& s_{1} \ldots s_{r}=1 \\
& s_{i+1} \ldots s_{r} s_{1} \ldots s_{i}=1 \\
& s_{1} \ldots s_{q+1}=s_{r} \ldots s_{q+2} \\
& s_{i+1} \ldots s_{j}=s_{i} \ldots s_{j-1} \\
& s_{i+1} \ldots s_{j-1} s_{j} \ldots s_{i}=1 .
\end{aligned}
$$

Rewriting given relation gives $\quad s_{i+1} \ldots s_{r} s_{1} \ldots s_{i}=1$.
LHS isn't reduced as RHS has length $q-1$ thus

Which gives us
If our final relation contains less than $r$ simple reflections then by inductive hypothesis it can also be derived in this manner and thus it is possible to delete from the relation to get the final relation of $s_{1} \ldots s_{i-1} s_{i+1} \ldots s_{j-1} s_{j+1} \ldots s_{r}=1$.

If our final relation contains $r$ simple reflections then we see that $i=1, j=q+1$ and we proceed as follows:

Starting with equation gives $s_{i+1} \ldots s_{j}=s_{i} \ldots s_{j-1}$.
After substitution we get $\quad s_{2} \ldots s_{q+1}=s_{1} \ldots s_{q}$.
Rewrite original relation $\quad s_{2} \ldots s_{r} s_{1}=1$.
If we now repeat the steps from before we will have a successful deletion unless $s_{3} \ldots s_{q+2}=s_{2} \ldots s_{q+1}$. If so, we can rewrite this producing

$$
s_{3}\left(s_{2} s_{3} \ldots s_{q+1}\right) s_{q+2} s_{q+1} \ldots s_{4}=1
$$

which has $r$ simple reflections just like our starting relation.
This will be successful with our original argument unless $s_{2} \ldots s_{q+1}=s_{3} s_{2} s_{3} \ldots s_{q}$, which together with $s_{2} \ldots s_{q+1}=s_{1} \ldots s_{q}$ would give us $s_{1}=s_{3}$. We can also permute the factors cyclically to reach a successful conclusion unless $s_{2}=s_{4}$. Inductively we can continue in this way until we get to the case

$$
\begin{aligned}
& s_{1}=s_{3}=\ldots s_{r-1} \\
& s_{2}=s_{4}=\ldots s_{r}
\end{aligned}
$$

Which then gives us that $s_{\alpha} s_{\beta} s_{\alpha} s_{\beta} \ldots s_{\alpha} s_{\beta}=1$ giving us a relation of the form $\left(s_{\alpha}, s_{\beta}\right)^{m(\alpha, \beta)}=1$ as desired.

### 3.4 Fundamental Domains ${ }^{[9]}$

We now look at some of the geometric implications of Coxeter groups when seen as reflections. First we look at parabolic subgroups.

### 3.4.1 Parabolic Subgroups

Definition. Let $\mathscr{W}$ be a Coxeter group generated by simple reflections from any fixed simple system $\Delta$. Let's look at the subgroups of $\mathscr{W}$ generated by subsets of $\Delta$. Let $\mathscr{S}$ be the set of simple reflections $s_{\alpha}$ where $\alpha \in \Delta$. Let $\mathscr{P}$ be a subset of $\mathscr{S}$ and define $\mathscr{W}_{\mathscr{P}}$ to be the group generated by all $s_{\alpha} \in \mathscr{P}$. $\mathscr{W}_{\mathscr{P}}$ is known as a parabolic subgroup. We also denote $\Delta_{\mathscr{P}}:=\left\{\alpha \in \Delta \mid s_{\alpha} \in \mathscr{P}\right\}, E_{\mathscr{P}}$ as the $\mathbb{R}$-span of $\Delta_{\mathscr{P}}$ in $E$, and $\Phi_{\mathscr{P}}:=\Phi \cap E_{\mathscr{P}}$.

We see easily that $\mathscr{W}_{\emptyset}=\{1\}$ and $\mathscr{W}_{\mathscr{S}}=\mathscr{W}$. We now state the following proposition without proof. A proof may be found in [9].

Proposition 3.4.1. Fix a simple system $\Delta$ and let $\mathscr{S}$ be the corresponding set of simple reflections. Let $\mathscr{P} \subset \mathscr{S}$.
(i) $\Phi_{\mathscr{D}}$ is a root system in $E$ (resp. $E_{\mathscr{P}}$ ) with simple system $\Delta_{\mathscr{P}}$ and with corresponding reflection group $\mathscr{W}_{\mathscr{P}}$ (resp. $\mathscr{W}_{\mathscr{P}}$ restricted to $E_{\mathscr{P}}$ ).
(ii) Viewing $\mathscr{W}_{\mathscr{P}}$ as a reflection group, with length function $\ell_{\mathscr{P}}$ relative to the simple system $\Delta_{\mathscr{P}}$, we have that $\ell=\ell_{\mathscr{P}}$ on $\mathscr{W}_{\mathscr{P}}$.
(iii) Define $\mathscr{W}^{\mathscr{P}}:=\{w \in \mathscr{W} \mid \ell(w s)>\ell(w) \forall s \in \mathscr{P}\}$. Now given $w \in \mathscr{W}$ there is a unique $u \in \mathscr{W}^{\mathscr{P}}$ and a unique $v \in \mathscr{W}_{\mathscr{P}}$ such that $w=u v$. Also, their lengths satisfy $\ell(w)=\ell(u)+\ell(v)$. Moreover, $u$ is the unique element of smallest length in the coset $w \mathscr{W}_{\mathscr{P}}$.

Part (c) of the previous proposition gives rise to a nice way of looking at the growth of $\mathscr{W}$ relative to its generating set $\mathscr{S}$. We first define a sequence $a_{n}:=\operatorname{Card}(\{w \in$ $\mathscr{W} \mid \ell(w)=n\})$.

Definition 3.4.2. A Poincaré polynomial is a polynomial with indeterminant $t$ defined as

$$
\mathscr{W}(t):=\sum_{n \geq 0} a_{n} t^{n}=\sum_{w \in \mathscr{W}} t^{\ell(w)}
$$

Since we saw in the previous proposition that $\ell_{\mathscr{P}}$ agrees with $\ell$ we see we can use the same equation for any subgroup $\mathscr{W}_{\mathscr{P}}$. We also see the following corollary to our previous proposition.

## Corollary 3.4.3.

$$
\mathscr{W}(t)=\mathscr{W}_{\mathscr{P}}(t) \mathscr{W}^{\mathscr{P}}(t)
$$

Proof. By the previous proposition we know that for each $w \in \mathscr{W}$ there exists a unique $u \in \mathscr{W}_{\mathscr{P}}$ and a unique $v \in \mathscr{W}^{\mathscr{P}}$ such that $\ell(w)=\ell(u)+\ell(v)$. Thus

$$
\begin{aligned}
W(t) & =\sum_{w \in \mathscr{W}} t^{\ell(w)} \\
& =\sum_{u \in \mathscr{W}_{\mathscr{P}}, v \in \mathscr{W}_{\mathscr{P}}} t^{\ell(u)+\ell(v)} \\
& =\sum_{u \in \mathscr{W}_{\mathscr{P}}, v \in \mathscr{W}_{\mathscr{P}}} t^{\ell(u)} t^{\ell(v)} \\
& =\sum_{u \in \mathscr{W}_{\mathscr{P}}} t^{\ell(u)} \sum_{v \in \mathscr{W}_{\mathscr{P}}} t^{\ell(v)} \\
& =\mathscr{W}_{\mathscr{P}}(t) \mathscr{W}^{\mathscr{P}}(t)
\end{aligned}
$$

### 3.4.2 Fundamental Domains

If we fix a positive system $\Pi$ which contains a unique simple system $\Delta$ we can associate with each hyperplane $H_{\alpha}$ the open half-spaces $A_{\alpha}$ and $-A_{\alpha}$ such that $A_{\alpha}:=\{\nu \in$
$E \mid(\nu, \alpha)>0 \quad \alpha \in \Delta\}$. A convex cone is a subset of a vector space that is closed under linear combinations with positive coefficients. We can create an open convex cone from our open half-spaces by defining $C:=\cap_{\alpha \in \Delta} A_{\alpha}$. We see this open and convex as each $A_{\alpha}$ is open and convex, and we see that each $A_{\alpha}$ is a linear combination of positive coefficients as we are in a positive system. We let $D$ be the closure of $C$. In other words $D:=\cap_{\alpha \in \Delta}\left(H_{\alpha} \cup A_{\alpha}\right)$. We can also rewrite $C$ and $D$ as

$$
\begin{aligned}
& C=\{\nu \in E \mid \quad(\nu, \alpha)>0 \quad \forall \alpha \in \Delta\} \\
& D=\{\nu \in E \mid \quad(\nu, \alpha) \geq 0 \quad \forall \alpha \in \Delta\}
\end{aligned}
$$

Definition 3.4.4. We say that a set $F$ is a fundamental domain for the action of $\mathscr{W}$ on $E$ if for each $\nu \in E$ there is a unique conjugate $\mu$ under $\mathscr{W}$ in $F$. The group $\{w \in \mathscr{W} \mid w \mu=\mu, \quad \mu \in E\}$ is called an isotropy group.

Theorem 3.4.5. Our closed convex cone $D$ is a fundamental domain.
Proof. (Rearranged from [9]) First we must show that such a conjugate exists. We therefore show that for each $\nu \in E$ there is a conjugation under $\mathscr{W}$ to $\mu \in D$. We first introduce a partial ordering for $E$. We say that $\lambda \leq \eta$ if and only if $\eta-\lambda$ is a linear combination of $\Delta$ with nonnegative coefficients. Consider the $\mathscr{W}$-conjugates $\mu$ of $\nu$ which satisfy $\nu \leq \mu$. This set at least contains $\nu$ and thus is nonempty. Choose a maximal element $\mu$ from this set. Now if $\alpha \in \Delta$, then $s_{\alpha}(\mu)$ is obtained from $\mu$ by subtracting a multiple of $\alpha$. If we remember our reflection formula we recall this multiple of $\alpha$ is just $\langle\mu, \alpha\rangle$. As this is another $\mathscr{W}$-conjugate of $\nu$, the maximality of $\mu$ forces $(\mu, \alpha) \geq 0$. This holds for all $\alpha \in \Delta$ so $\mu \in D$ as we wanted.

Now we just need to show that only one such $\mu$ exists. Only one such $\mu$ exists if no pair of distinct elements of $D$ can be $\mathscr{W}$-conjugate, or in other words, the isotropy group of $\nu \in E$ is trivial. To do this we must show that if $w \nu=\mu$ for $\nu, \mu \in D$ then $\nu=\mu$ and as $D$ contains $C$ the result will follow.

We proceed by induction on $\ell(w)=\pi(w)$. If $\pi(w)=0$ then $w=1$ and $\ell(w)=0$ giving us our initial step. Now assume that $\pi(w)>0$. Then $w$ must send some simple root $\alpha$ to a negative root otherwise $w \Delta$ and therefore $w \Pi$ would consist of positive roots. We thus know from lemma 3.3.6 that $\pi\left(w s_{\alpha}\right)=\pi(w)-1$. Therefore, since $\nu, \mu \in D$ with $w \alpha<0$ we have that $0 \geq(\mu, w \alpha)=\left(w^{-1} \mu, w^{-1} w \alpha\right)=(\nu, \alpha) \geq 0$. This forces $(\nu, \alpha)$ to be 0 and $s_{\alpha} \nu=\nu$. We therefore have $w s_{\alpha} \nu=\mu$ and by induction we


Figure 3.1: Fundamental Chamber of $A_{2}$
know that $\mu=\nu$ and $w s_{\alpha}$ is a product of simple reflections which fix $\nu$. Therefore $w$ is such a product as well. Therefore the isotropy group of $\nu$ is trivial and our proof is complete.

A chamber is just the open convex cone in $E$ associated with a simple system $\Delta$ whose points all have trivial isotropy groups in $\mathscr{W}$. Our choice of the letter $C$ earlier was due to those sets being chambers. Each chamber also has corresponding walls which are defined to be the hyperplanes $H_{\alpha}$ such that $\alpha \in \Delta$. Each wall has a positive and negative side with the chamber $C$ lying on its positive side.

By looking at figure 3.1 we can see that our hyperplanes $H_{\alpha}$ and $H_{\beta}$ form walls for the shaded in region which is the chamber with respect to our simple system $\Delta=\{\alpha, \beta\}$. This particular chamber is known as the fundamental chamber.

We can relook at roots now and see them as the vectors which are orthogonal to some wall of $C$ and are positively directed. We also easily notice that the angle between any two walls of a chamber must be an angle of the form $\pi / k$ for $k$ a positive integer greater than 1.

### 3.5 Irreducible Root Systems ${ }^{[8]}$

We say that $\Phi$ is an irreducible root system if it cannot be partitioned into the union of two proper subsets such that each root in one is orthogonal to each root in the other. We say $\Phi$ is reducible if $\Phi=\Phi_{1} \cup \Phi_{2}$ such that $\left(\Phi_{1}, \Phi_{2}\right)=0$. As simple systems generate root systems we might then wonder whether the irreducibility of $\Phi$ is related to the ability to partition its simple system $\Delta$. It turns out they are very closely related.

Theorem 3.5.1. $\Phi$ is irreducible if and only if $\Delta$ cannot be partitioned into the union of two proper subsets such that each root in one is orthogonal to each root in the other. Proof. [8] First we show the converse. We first suppose contrarily that $\Phi$ is reducible. Let $\Phi=\Phi_{1} \cup \Phi_{2}$ such that $\left(\Phi_{1}, \Phi_{2}\right)=0$. Unless $\Delta$ is wholly in $\Phi_{1}$ or $\Phi_{2}$ then $\Delta$ has a similar partition which isn't allowed. Thus $\Delta \subset \Phi_{1}$ which implies that $\left(\Delta, \Phi_{2}\right)=0$ and thus $\left(E, \Phi_{2}\right)=0$ as $\Delta$ spans $E$ giving us a contradiction.

Now we assume that $\Phi$ is irreducible. Suppose that $\Delta$ is reducible and thus $\Delta=$ $\Delta_{1} \cup \Delta_{2}$ such that $\left(\Delta_{1}, \Delta_{2}\right)=0$. Each root is conjugate to a simple root as shown in the proof of theorem 3.4.5. Therefore $\Phi=\Phi_{1} \cup \Phi_{2}$ with each $\Phi_{i}$ being the set of roots having a conjugate in $\Delta_{i}$. Now $(\alpha, \beta)=0$ implies that $s_{\alpha} s_{\beta}=s_{\beta} s_{\alpha}$ and as $\mathscr{W}$ is generated by the $s_{\alpha}$ the reflection formula tells us that each root in $\Phi_{i}$ is a linear combination of the elements of $\Delta_{i}$. Thus $\Phi_{i}$ lies in $E_{i}$ which is a subspace of $E$ spanned by $\Delta_{i}$. Therefore $\left(\Phi_{1}, \Phi_{2}\right)=0$ which implies that either $\Phi_{1}=\emptyset$ or $\Phi_{2}=\emptyset$ and therefore $\Delta_{1}=\emptyset$ or $\Delta_{2}=\emptyset$.

## Chapter 4

## Classification of Root Systems

Dynkin figured out that by looking at root systems one can classify all semisimple Lie algebras. We start with Coxeter graphs and progress to proving the classification theorem for both Dynkin diagrams and Coxeter groups.

### 4.1 Coxeter Graphs ${ }^{[9]}$

We start by introducing a type of graph called a Coxeter graph in order to help us visualise roots better.

Definition 4.1.1 (Coxeter Graph). A Coxeter graph $\Gamma$ is a graph of a Coxeter group $\mathscr{W}$ in which for each $s \in \mathscr{S}$ we have a vertex, and between any two vertices $s, s^{\prime}$ we place a labelled edge whenever $m\left(s, s^{\prime}\right) \geq 3$ and label it with the value of $m\left(s, s^{\prime}\right)$ (omitting the label when $m\left(s, s^{\prime}\right)=3$ ).

We say that a Coxeter system $(\mathscr{W}, \mathscr{S})$ is irreducible if its associated Coxeter graph is connected. As an abuse of language we say that $\Phi$ is irreducible as well. As we saw $\Phi$ is irreducible if and only if $\Delta$ cannot be partitioned in two orthogonal sets. It should therefore be obvious by our definitions that $\Phi$ is irreducible if and only if its Coxeter graph is connected.

We allow each Coxeter graph to have an associated symmetric $n \times n$ matrix $A$ called a Cartan matrix. We use the definition as seen in [3] with a slight alteration.

A Cartan matix has entries

$$
a_{i, j}:= \begin{cases}-2 \frac{\left\|\alpha_{i}\right\|}{\left\|\alpha_{j}\right\|} \cos \frac{\pi}{m\left(s_{\alpha_{i}}, s_{\alpha_{j}}\right)} & \text { if } i \neq j \\ 2 & \text { if } i=j\end{cases}
$$

for $s_{i}, s_{j} \in \mathscr{S}$.
We say that $A$ is positive definite if $x^{T} A x>0$ for all $x \neq 0$. We say that $A$ is positive semidefinite if $x^{T} A x \geq 0$ for all $x$. Finally we say that $A$ is of positive type if $A$ is either positive definite or positive semidefinite. We notice that $A$ is positive definite (resp. semidefinite) if and only if the principal minors of $A$ have determinant greater than 0 (resp. nonnegative). We say that a real $n \times n$ matrix $A$ is indecomposable if there is no partition of the index set into nonempty subsets $I, J$ such that $a_{i, j}=0$ whenever $i \in I, j \in J$.

We define a subgraph of a Coxeter graph $\Gamma$ to be a graph which omits some vertices and any edges adjacent to the removed vertices, or decreases the labels by 1 (or both). It should be clear that a Cartan matrix is indecomposable if and only if its Coxeter graph is connected.

### 4.2 Crystallographic Root Systems ${ }^{[5],[8]}$

In this section we will exclusively work in crystallographic root systems.
Lemma 4.2.1 (Finiteness Lemma). Let $\Phi$ be a crystallographic root system for the Euclidean space $E$. Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm \beta$. Then

$$
\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle \in\{0,1,2,3\} .
$$

Proof. (Based on [5]) Since we are only looking at crystallographic root systems we know the product must be an integer. From earlier discussions on Euclidean space, we recall that $\cos \theta=\frac{(v, w)}{\|v\|\|w\|}$. Now we see that

$$
\begin{aligned}
\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle & =\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \frac{2(\beta, \alpha)}{(\beta, \beta)} \\
& =\frac{2\|\alpha\|\|\beta\| \cos \theta}{\|\alpha\|\|\alpha\|} \frac{2\|\beta\|\|\alpha\| \cos \theta}{\|\beta\|\|\beta\|} \\
& =4 \cos ^{2} \theta
\end{aligned}
$$

And as $0 \leq \cos ^{2} \theta \leq 1$ we see that $0 \leq 4 \cos ^{2} \theta \leq 4$.

To make our bound stricter we assume that $\cos ^{2} \theta=1$ and notice that $\theta=\pi$. This implies that $\alpha=-\beta$ which contradicts our assumptions and thus $0 \leq 4 \cos ^{2} \theta<4$, giving us our lemma.

From here we can use the finiteness lemma to classify all potential possibilities of multiplying two Cartan integers. Thus given any two arbitrary roots $\alpha, \beta \in \Phi$ we see the following:

| $\langle\alpha, \beta\rangle$ | $\langle\beta, \alpha\rangle$ | $\theta$ | $\frac{(\beta, \beta)}{(\alpha, \alpha)}=\frac{\\|\beta\\|^{2}}{\\|\alpha\\|^{2}}$ |
| ---: | :---: | :---: | :--- |
| 0 | 0 | $\pi / 2$ | undefined |
| 1 | 1 | $\pi / 3$ | 1 |
| -1 | -1 | $2 \pi / 3$ | 1 |
| 1 | 2 | $\pi / 4$ | 2 |
| -1 | -2 | $3 \pi / 4$ | 2 |
| 1 | 3 | $\pi / 6$ | 3 |
| -1 | -3 | $5 \pi / 6$ | 3 |

This allows us to start drawing out what these root spaces might look like. We'll give examples of all 1 and 2 dimensional root systems and also do a single 3 dimensional example.

Examples 4.2.2. (i) First let's start off with a 1-dimensional root space. As we only have one vector we see that there is only one possibility.


Figure 4.1: $A_{1}$
(ii) We now look at root systems with dimension two. We see that in each case we can get a description of $\Phi$ by looking at $\alpha, \beta \in \Delta$. From the finiteness lemma we see that the angles between our two vectors must be either $\pi / 2,2 \pi / 3,3 \pi / 4$ or $5 \pi / 6(\langle\beta, \alpha\rangle \leq 0$ by corollary 3.2.4). Thus we construct our pictures as follows.


Figure 4.2: Dimension two root system
(iii) We lastly look at a three dimensional case. Let $\alpha, \beta, \gamma \in \Delta$ be such that the angles between them are set as $\theta(\alpha, \beta)=\frac{3 \pi}{4}, \theta(\beta, \gamma)=\frac{2 \pi}{3}$, and $\theta(\alpha, \gamma)=\frac{\pi}{2}$. This gives us the structure below.


Figure 4.3: $B_{3}$

### 4.3 Dynkin Diagrams ${ }^{[8]}$

Looking just at crystallographic root systems, we can also make our definitions of a Cartan matrix and a Coxeter graph slightly simpler.

Definition 4.3.1. Let $\Delta$ be a simple system for a root system $\Phi$ where $\Delta$ has $n$ roots. The Cartan matrix of $\Phi$ is the $n \times n$ matrix such that for $\alpha_{i}, \alpha_{j} \in \Delta$ we have entries

$$
a_{i, j}=\left\langle\alpha_{i}, \alpha_{j}\right\rangle
$$

Note that $\left\langle\alpha_{i}, \alpha_{j}\right\rangle=\left\langle s_{\beta}\left(\alpha_{i}\right), s_{\beta}\left(\alpha_{j}\right)\right\rangle$ for $\beta \in \Delta$.

Example 4.3.2. As an example of a Cartan matrix we look at $A_{2}$. We see that $A_{2}$ has dimension two and therefore $\Delta$ has two elements. Call them $\alpha$ and $\beta$. We know that $\langle\alpha, \alpha\rangle=\frac{2(\alpha, \alpha)}{(\alpha, \alpha)}=2,\langle\beta, \beta\rangle=2$ and that $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle=1$. Also noting that $\alpha, \beta \in \Delta$ implies $\langle\beta, \alpha\rangle<0$, we get a Cartan matrix of $\left(\begin{array}{cc}2 & -1 \\ -1 & 2\end{array}\right)$ for $A_{2}$ (with ordered basis $\{\alpha, \beta\}=\Delta$ ).

We saw earlier that a Coxeter graph can have an associated Cartan matrix. We can use the Cartan matrix to construct a graph that is very similar to a Coxeter graph.

Definition 4.3.3 (Dynkin Diagram). Let the Dynkin diagram of a root system $\Phi$ be the connected directed graph with $n$ vertices where the $i$ th and $j$ th vertex $(i \neq j)$ are joined by $\left\langle\alpha_{i}, \alpha_{j}\right\rangle\left\langle\alpha_{j}, \alpha_{i}\right\rangle$ edges and such that if any two roots have different lengths then we draw an arrow from the longer root to the shorter root.

Example 4.3.4. So from our example of $A_{2}$ with Cartan matrix $\left(\begin{array}{cc}2 & -1 \\ -1 & 2\end{array}\right)$ we see that we can construct a Dynkin diagram


We see that a Dynkin diagram is the same thing as a Coxeter graph such that no arrows are present, a double edge in a Dynkin diagram is the same as an edge labelled with a 4 in a Coxeter graph and a triple edge as an edge labelled with a 6. By abuse of language we shall call a Dynkin diagram without any arrows a Coxeter graph even if we have multiple edges instead of labelled edges.

By the finiteness lemma we wonder whether we might be able to classify all different types of Dynkin diagrams and it turns out we can.

### 4.4 Classification

We will give a full classification of Dynkin diagrams using the original proof as outlined by Dynkin. We will then give a full classification of Coxeter graphs for all finite Coxeter groups as outlined by Humphreys.

### 4.4.1 Classification of Dynkin Diagrams ${ }^{[4]}$

Theorem 4.4.1. If $\Phi$ is crystallographic and is generated by a simple system $\Delta$ and $\lambda$ is an arbitrary positive number representing the length of a vector then the Dynkin diagram must be one of the following:


We will break down this theorem into its basic parts. As $\Phi$ is crystallographic the finiteness lemma says that for any $\alpha, \beta \in \Delta, \theta(\alpha, \beta) \in\left\{\frac{\pi}{2}, \frac{2 \pi}{3}, \frac{3 \pi}{4}, \frac{5 \pi}{6}\right\}$ and therefore $\cos \theta(\alpha, \beta) \in\left\{0,-\frac{1}{2},-\frac{\sqrt{2}}{2},-\frac{\sqrt{3}}{2}\right\}$.

Lemma 4.4.2. If $\Delta$ consists of 3 vectors then its Coxeter graph is one of the following two options:


Proof. (Outline provided by [4]) We know from geometry that the angles between any three linearly independent vectors $\alpha_{1}, \alpha_{2}, \alpha_{3}$ sum to strictly less than $2 \pi$. Also if $\alpha_{1}, \alpha_{2}, \alpha_{3} \in \Delta$ then each of $\theta\left(\alpha_{1}, \alpha_{2}\right), \theta\left(\alpha_{1}, \alpha_{3}\right)$, and $\theta\left(\alpha_{2}, \alpha_{3}\right)$ are in $\left\{\frac{\pi}{2}, \frac{2 \pi}{3}, \frac{3 \pi}{4}, \frac{5 \pi}{6}\right\}$.

We can't have a cycle containing all three vertices as then this would imply that each two vectors have an angle of at least $\frac{2 \pi}{3}$ between them which would give us $2 \pi>\frac{2 \pi}{3}+\frac{2 \pi}{3}+\frac{2 \pi}{3}=2 \pi$. Also, it can easily be seen that the graph on the left is possible. Therefore that is the only possible Coxeter graph with three vertices and all single edges.

Suppose we have a double edge between two arbitrary vertices. Without loss of generality suppose the double edge is between $\alpha_{1}$ and $\alpha_{2}$. Then $\theta\left(\alpha_{1}, \alpha_{2}\right)=\frac{3 \pi}{4}$ and as $2 \pi>\frac{2 \pi}{3}+\frac{3 \pi}{4}+\frac{\pi}{2}$ we know that our right-hand graph is possible. We also know we
can't have two double edges else we would get $2 \pi>\frac{3 \pi}{4}+\frac{3 \pi}{4}+\frac{\pi}{2}=2 \pi$. Therefore the right-hand graph is the only possible graph with three vertices and at least one double edge.

All that is left to show is that we cannot have a graph with a triple edge. Indeed, suppose there were a triple edge between $\alpha_{1}$ and $\alpha_{2}$. As the graph is required to be connected this forces minimally a single edge between $\alpha_{2}$ and $\alpha_{3}$. Counting our angles we have we have $2 \pi>\frac{5 \pi}{6}+\frac{2 \pi}{3}+\frac{\pi}{2}=2 \pi$ which cannot happen.

Thus our two graphs are the only possible constructions.

Lemma 4.4.3 (Reduction lemma). Reducing a Coxeter graph to a subgraph by either removing a vertex (and adjacent edges) or reducing the number of edges when more than one edge exists, you still get a valid Coxeter graph.

Proof. As the procedure in the lemma takes a connected graph and creates a connected graph, this graph must be associated with some simple system and is thus a Coxeter graph itself.

Lemma 4.4.4. The only Coxeter graph that contains a triple edge is a Coxeter graph with two vertices.

Proof. (Outline provided by [4]) If a graph has only one vertex then it cannot have a triple edge to itself as $\langle\alpha, \alpha\rangle=2$ which implies that there is no edge to itself.

Now suppose that contrarily there was a Coxeter graph that contained a triple edge with more than two vertices. By the reduction lemma we can keep removing vertices while keeping the triple edge in tact. As Coxeter graphs are finite we can eventually do this until we hit three vertices. But then by the lemma 4.4.2 if a Coxeter graph has three vertices it cannot have a triple edge. Therefore we have reached a contradiction and our lemma is proved.

Lemma 4.4.5. The Coxeter graph of $\Delta$ cannot have the following forms:




Proof. (Outline provided by [4]) Let $\Delta=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$. We let $\beta_{i}=\lambda_{i} \alpha_{i}$ for $1 \leq$ $i \leq n$ where $\lambda_{i} \in \mathbb{R}$ and are nonzero. Since the vectors in $\Delta$ are linearly independent, we see that $\sum_{i=1}^{n} \beta_{i}=\sum_{i=1}^{n} \lambda_{i} \alpha_{i} \neq 0$. Letting $(-,-)$ be the positive definite symmetric bilinear form developed in section 2.8, we see that $\sum_{i=1}^{n} \sum_{j=1}^{n}\left(\beta_{i}, \beta_{j}\right)=$ $\left(\sum_{i=1}^{n} \beta_{i}, \sum_{j=1}^{n} \beta_{j}\right)=\left(\sum_{i=1}^{n} \beta_{i}, \sum_{i=1}^{n} \beta_{i}\right)>0$ as $(-,-)$ is positive definite and the summations are not 0 .

We can reach a contradiction if we can make the following summation be true

$$
\begin{equation*}
\sum_{i=1}^{n} \sum_{j=1}^{n}\left(\beta_{i}, \beta_{j}\right) \leq 0 \tag{4.1}
\end{equation*}
$$

If we label each vertex below with ( $\beta_{i}, \beta_{i}$ ) and each edge (or pair of edges) above with ( $\beta_{i}, \beta_{j}$ ) we can easily see that (4.1) holds. The labelling that satisfies our equation is



$I_{4} \quad \underset{2}{\mathrm{O}} \underset{8}{-2} \mathrm{O} \xlongequal[9]{-6}{ }_{9}^{-6} \mathrm{O}_{4}^{-3} \mathrm{O}_{1}^{-1}$



Figure 4.4

Thus we see that (4.1) holds for each of the graphs and therefore we get a contradiction for each one and thus none of these graphs can be a Coxeter graph.

Example. As a demonstration of the above, let's break down where the numbers are coming from and whether they do satisfy all of the requirements. We will look at $I_{2}$ as it has components present in almost every other graph and thus our methods can be used in most of the other ones as well. Let's first look at two labellings of the graph side by side. The graph on the left in figure 4.4 labels each vertex with the vectors in $\Delta$ it is associated with. The one on the right in figure 4.4 is the values of the positive definite symmetric bilinear form as seen in the proof above. We want to show this graph is impossible to have.

Now, we note that the two figures above imply:

$$
\begin{aligned}
& \left(\beta_{i}, \beta_{i}\right)= \begin{cases}2 & \text { if } i=1 \\
1 & \text { if } i \in\{n-1, n\} \\
4 & \text { otherwise }\end{cases} \\
& \left(\beta_{i}, \beta_{i+1}\right)= \begin{cases}-2 & \text { if } 1 \leq i \leq n-3 \\
-1 & \text { if } i=n-2\end{cases} \\
& \left(\beta_{n-2}, \beta_{n}\right)=-1
\end{aligned}
$$

Now we just need to verify if these numbers make sense. As $\left(\beta_{i}, \beta_{i}\right)$ must be a positive integer, the first set of equations work properly. To see the second two sets of equations work we first recall that $\cos ^{2} \theta\left(\beta_{i}, \beta_{j}\right)=\frac{\left(\beta_{i}, \beta_{j}\right)\left(\beta_{j}, \beta_{i}\right)}{\left(\beta_{i}, \beta_{i}\right)\left(\beta_{j}, \beta_{j}\right)}$. We will look at three cases:
(i) The vertices labelled 4 with a single edge between them labelled -2 . We see this works out as we must have $\cos ^{2}\left(\frac{2 \pi}{3}\right)=\left(\frac{-1}{2}\right)^{2}=\frac{1}{4}=\frac{-2 \cdot-2}{4 \cdot 4}$ which is valid.
(ii) Next we look at the two vertices labelled 1 with adjacent vertex labelled with a 4. The edge between these two vertices is a single edge with label -1 . We see this works out as we must have $\cos ^{2}\left(\frac{2 \pi}{3}\right)=\left(\frac{-1}{2}\right)^{2}=\frac{1}{4}=\frac{-1 \cdot-1}{1 \cdot 4}$ which is valid.
(iii) Finally we look at the double edge. The two vertices are labelled 4 and 2 and the double edge has a label of -2 . We see this works out as we must have $\cos ^{2}\left(\frac{3 \pi}{4}\right)=\left(\frac{-\sqrt{2}}{2}\right)^{2}=\frac{1}{2}=\frac{-2 \cdot-2}{2 \cdot 4}$ which is valid.

The last thing to ensure is that our summations hold as well. We see that

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{n}\left(\beta_{i}, \beta_{j}\right) & =\sum_{i=1}^{n}\left(\beta_{i}, \beta_{i}\right)+\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n}\left(\beta_{i}, \beta_{j}\right) \\
& =(4 \cdot(n-3)+2+1 \cdot(2))+2(-2(n-2)-1(2)) \\
& =(4 n-12+2+2)+2(-2 n+4-2) \\
& =4 n-4-4 n+8-4 \\
& =0
\end{aligned}
$$

which contradicts that it must be strictly greater than 0 .
Lemma 4.4.6. An arbitrary simple system $\Delta$ must have an associated Coxeter graph of one of the following types
$I \quad=0$
$I I^{1}$


$I I I^{4}$

$I I I^{5}$


Proof. (Outline provided by [4]) We've already proved that the only Coxeter graph with a triple edge is $I$.

Let's next suppose that our Coxeter graph $\Gamma$ has a double edge. From the previous lemma and the reduction lemma we can deduce
(i) By $I_{1}, \Gamma$ cannot contain a second double edge.
(ii) By $I_{2}$, no vertex in $\Gamma$ can have three or more adjacent vertices.
(iii) By $I_{3}$, there are at least two vertices in $\Gamma$ with only one adjacent vertex.
(iv) By $I_{4}$, if the vertices attached by the double edge each have two adjacent edges then we must have graph $I I^{2}$.

From the above points we see that the only possible Coxeter graphs with a double edge are $I I^{1}$ and $I I^{2}$.

Finally let's suppose that our Coxeter graph contains only single edges. We define an end vertex as a vertex with only one adjacent edge. We say a chain from a vertex to an end vertex has length $n$ where $n$ is the number of verticies from the start vertex to the end vertex not counting the start vertex. As an example, the graph below gives us a chain of length 4 for vertex $\alpha$ to end vertex $\beta$.


From the previous lemma and the reduction lemma we can deduce
(i) By $I I_{1}, \Gamma$ can only have one vertex with three adjacent vetices.
(ii) By $I I_{2}$, If $\Gamma$ has a vertex with three adjacent vertices, at least one of those vertices must be an end vertex.
(iii) By $I I_{3}, \Gamma$ must contain at least two vertices with only one adjacent vertex.
(iv) By $I I_{4}$, If $\Gamma$ has a vertex with three adjacent vertices, then only one chain from that vertex may have length greater than 2.
(v) By $I I_{5}$, If $\Gamma$ has a vertex with three adjacent vertices, then no chain from that vertex can have length greater than 4.

From these points, we see that the only graphs that satisfy our requirements are the ones shown.

Proof of theorem. (Outline provided by [4]) As we saw, the only Coxeter graphs we are allowed are the ones shown in lemma 4.4.6. All we need to do is provide the lengths thus giving us the Dynkin diagrams. Let's suppose we choose two arbitrary roots $\alpha, \beta \in \Delta$.

First let's suppose that the vertices associated with $\alpha$ and $\beta$ have a single edge, thus $\theta(\alpha, \beta)=\frac{2 \pi}{3}$. Therefore

$$
\frac{2(\alpha, \beta)}{(\beta, \beta)} \cdot \frac{2(\beta, \alpha)}{(\alpha, \alpha)}=4 \cos ^{2} \theta(\alpha, \beta)=1
$$

Which then implies that

$$
\frac{2(\alpha, \beta)}{(\beta, \beta)}=\frac{2(\beta, \alpha)}{(\alpha, \alpha)}=-1
$$

and therefore we must have $(\alpha, \alpha)=(\beta, \beta)$.
If the vertices associated with $\alpha$ and $\beta$ have a double edge then $\theta(\alpha, \beta)=\frac{3 \pi}{4}$. Therefore

$$
\frac{2(\alpha, \beta)}{(\beta, \beta)} \cdot \frac{2(\beta, \alpha)}{(\alpha, \alpha)}=4 \cos ^{2} \theta(\alpha, \beta)=2
$$

Which then implies that

$$
\frac{2(\alpha, \beta)}{(\beta, \beta)}=\frac{2(\beta, \alpha)}{2(\alpha, \alpha)}=-1
$$

and therefore we must have $2(\alpha, \alpha)=(\beta, \beta)$.
Finally if the vertices associated with $\alpha$ and $\beta$ have a triple edge then $\theta(\alpha, \beta)=\frac{5 \pi}{6}$. Therefore

$$
\frac{2(\alpha, \beta)}{(\beta, \beta)} \cdot \frac{2(\beta, \alpha)}{(\alpha, \alpha)}=4 \cos ^{2} \theta(\alpha, \beta)=3 .
$$

Which then implies that

$$
\frac{2(\alpha, \beta)}{(\beta, \beta)}=\frac{2(\beta, \alpha)}{3(\alpha, \alpha)}=-1
$$

and therefore we must have $3(\alpha, \alpha)=(\beta, \beta)$.
Furthermore, we can draw an arrow from a vertex with a larger label to a vertex with a smaller label thus giving our Dynkin diagrams as described.

### 4.4.2 Classification of Coxeter Graphs ${ }^{[9]}$

We now turn our attention to the classification of all Coxeter graphs.

Theorem 4.4.7. The only connected Coxeter graphs of positive type are

$$
A_{n}(n \geq 1)
$$

$$
\mathrm{O}-\mathrm{O}-\mathrm{O}^{----\mathrm{O}} \mathrm{O}-\mathrm{O}
$$

$F_{4}$

$B_{n}(n \geq 2)$

$\mathrm{H}_{3}$


$H_{4}$

$E_{6}$

$E_{8}$


Lemma 4.4.8. The graphs in theorem (4.4.7) are all positive definite graphs.
Proof. (Outlined by [9]) We do this by induction on $n$ (the number of vertices). Remember that a matrix $A$ is positive definite if and only if its principal minors are positive.

If $n \leq 2$ then we can check things directly. We can do this by looking at the Cartan matrix $A$ for $I_{2}(m)$ :

$$
\left(\begin{array}{cc}
2 & -\sqrt{2 m} \cos (\pi / m) \\
-\frac{4 \cos (\pi / m)}{\sqrt{2 m}} & 2
\end{array}\right)
$$

We easily see that

$$
\begin{aligned}
\operatorname{det}(A) & =\left(2 \cdot 2-\left(-\sqrt{2 m} \cos (\pi / m) \cdot-\frac{4 \cos (\pi / m)}{\sqrt{2 m}}\right)\right. \\
& =4\left(1-\cos ^{2}(\pi / m)\right) \\
& =4\left(\sin ^{2}(\pi / m)\right) \\
& >0
\end{aligned}
$$

Now let's look at $n \geq 3$. Now looking at the figure in theorem (4.4.7) we see that we can number the vertices from left to right so that the first vertex is joined by an edge to only one other vertex (in our case labelled $n-1$ ), and this edge having a label either 3 or 4 . In this way we can define $d_{i}$ to be the determinant for the bottom right $i \times i$ submatrix of $A$. Now expanding $\operatorname{det}(A)$ along the first rows shows us that $\operatorname{det}(A)=2 d_{n-1}-c d_{n-2}$ where $c=1$ if our edge is labelled 3 and $c=2$ if our initial edge is labelled 4 . In this way, we can verify the determinant of each matrix.

We do this in a case by case basis. For each case let $A$ be the associated Cartan matrix.
$A_{n}$ We claim that for $A_{n} \operatorname{det}(A)=n+1$. We do this by induction. For $n=1$ we know we have a $1 \times 1$ with entry 2 and therefore we see that $\operatorname{det}(A)=2$. Now suppose this is true $n \geq 1$. We know that $\operatorname{det}(A)=2 d_{n-1}-d_{n-2}=$ $2(n)-(n-1)=2 n-n+1=n+1$ as we desired.
$B_{n}$ We claim that for $B_{n} \operatorname{det}(A)=2$. For $B_{2}$ we have $A=\left(\begin{array}{cc}2 & -2 \\ -1 & 2\end{array}\right)$ and therefore $\operatorname{det}(A)=(4-2)=2$. Now by induction suppose this is true for $n>2$. As we are removing a vertex with edge having label 4 we see that $\operatorname{det}(A)=2 d_{n-1}-2 d_{n-2}$
and our principal minors are $A_{n-1}$ and $A_{n-2}$ thus giving us $\operatorname{det}(A)=2 A_{n-1}-$ $2 A_{n-2}=2(n)-2(n-1)=2$. And thus our claim is true.
$D_{n}$ We claim that for $D_{n} \operatorname{det}(A)=4$. We do this by induction as before. For $n=4$ we know that the matrix $A=\left(\begin{array}{cccc}2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2\end{array}\right)$ which gives us $\operatorname{det}(A)=4$ as required. Now let $n>4$. Then for $D_{n+1}$ we have our principal minors as $D_{n-1}$ and $D_{n-2}$ and thus $\operatorname{det}(A)=2\left(d_{n-1}\right)-\left(d_{n-2}\right)=2 \cdot 4-4=4$ as required. Note that when $n=5$ our second principal minor is in fact $A_{3}$ but the determinant of $A_{3}$ is 4 as well.
$E_{6}$ We see that $E_{6}$ comes from having principal minors from graphs of type $D_{5}$ and $A_{4}$. Thus we get $\operatorname{det}(A)=2(4)-5=3$.
$E_{7}$ We see that $E_{7}$ comes from having principal minors from graphs of type $E_{6}$ and $D_{5}$. Thus we get $\operatorname{det}(A)=2(3)-4=2$.
$E_{8}$ We see that $E_{8}$ comes from having principal minors from graphs of type $E_{7}$ and $E_{6}$. Thus we get $\operatorname{det}(A)=2(2)-3=1$.
$F_{4}$ We see that $F_{4}$ comes from having principal minors from graphs of type $B_{3}$ and $A_{2}$. Thus we get $\operatorname{det}(A)=2(2)-3=1$.
$H_{3}$ We see that $H_{3}$ comes from having principal minors from graphs of type $I_{2}(5)$ and $A_{1}$. Thus we get $\operatorname{det}(A)=2\left(4 \sin ^{2}(\pi / 5)\right)-2=8\left(1-\left(\frac{1+\sqrt{5}}{4}\right)^{2}\right)-2=$ $6-8\left(\frac{6+2 \sqrt{5}}{16}\right)=6-3+\sqrt{5}=3-\sqrt{5}$.
$H_{4}$ We see that $H_{4}$ comes from having principal minors from graphs of type $H_{3}$ and $I_{2}(5)$. Thus we get $\operatorname{det}(A)=2(3-\sqrt{5})-4 \sin ^{2}(\pi / 5)=6-\sqrt{5}-4+4\left(\frac{1+\sqrt{5}}{4}\right)^{2}=$ $2-\sqrt{5}+\frac{3-\sqrt{5}}{2}=\frac{7-3 \sqrt{5}}{2}$.

We therefore see that every determinant is positive and therefore the associated Coxeter graph is a positive definite graph.

Lemma 4.4.9. The following graphs are all positive semidefinite graphs.


Proof. (Outlined by [9]) In order to show these graphs are positive semidefinite we want to show that the determinant of each one is equal to 0 . As before we let $A$ be the associated Cartan matrix and recall that $\operatorname{det}(A)=2 d_{n-1}-c d_{n-1}$.
$\widetilde{A}_{1}$ Using the matrix for $I_{2}(m)$ we see that

$$
\operatorname{det}(A)=\lim _{m \rightarrow \infty} \operatorname{det}\left(\begin{array}{cc}
2 & -\sqrt{2 m} \cos (\pi / m) \\
-\frac{4 \cos (\pi / m)}{\sqrt{2 m}} & 2
\end{array}\right)=\operatorname{det}\left(\begin{array}{cc}
2 & -\sqrt{2 m} \\
-\frac{4}{\sqrt{2 m}} & 2
\end{array}\right)=4-4=0
$$

$\widetilde{A}_{n}$ As $\widetilde{A}_{n}$ forms a loop, each row will have a 2 and two -1 (one for each single edge) and therefore the sum of all the rows will equal 0 . Therefore the determinant is 0.
$\widetilde{B}_{2}$ We know that our matrix is represented by $A=\left(\begin{array}{ccc}2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2\end{array}\right)$. Therefore we have $\operatorname{det}(A)=8-4-4=0$.
$\widetilde{B}_{n}$ We solve this by induction. For $n=3$ we get $A=\left(\begin{array}{cccc}2 & -2 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2\end{array}\right)$ and therefore we get $\operatorname{det}(A)=2(8-4-2)-1(4)=4-4=0$. Now our inductive step shows us that if $\operatorname{det}(A)=0$ for some $n>3$ then for $\widetilde{B}_{n+1}$ we have that $\operatorname{det}(A)=2 d_{n-1}-$ $2 d_{n-2}$ as the removed edge has a label 4. Also our two principal minors are the graphs $D_{n}$ and $D n-1$ thus giving us $\operatorname{det}(A)=2 d_{n-1}-2 d_{n-2}=2(4)-2(4)=0$ and therefore our claim is true.
$\widetilde{C}_{n}$ For $\widetilde{C}_{3}$ we have $A=\left(\begin{array}{cccc}2 & -2 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2\end{array}\right)$ and therefore $\operatorname{det}(A)=2(8-2-4)+2(-4+$ $2)=4+-4=0$. Now by induction suppose this is true for $n>3$. Then for
$\widetilde{C}_{n+1}$ we have principal ideals of $B_{n}$ and $B_{n-1}$ and thus $\operatorname{det}(A)=2 d_{n-1}-d_{n-2}=$ $2(2)-2(2)=0$ as our initial edge has label 4 .
$\widetilde{D}_{n}$ We show the determinants are 0 by induction as before. For $n=4$ we know that the matrix $A=\left(\begin{array}{ccccc}2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2\end{array}\right)$ which gives us $\operatorname{det}(A)=0$ as required. Now let $n>4$. $\widetilde{D}_{n+1}$ has a slightly different matrix than most, but by induction we can see that the Cartan matrix has determinant $\operatorname{det}(A)=2 \operatorname{det}\left(A_{D_{n+1}}\right)-$ $2 \operatorname{det}\left(A_{D_{n-1}}\right)=2(4)-2(4)=0$ as required, where $A_{D_{n}}$ is the Cartan matrix associated with type $D_{n}$.
$\widetilde{E}_{6}$ We see that $\widetilde{E}_{6}$ comes from having principal minors from graphs of type $E_{6}$ and $A_{5}$. Thus we get $\operatorname{det}(A)=2(3)-6=0$.
$\widetilde{E}_{7}$ We see that $\widetilde{E}_{7}$ comes from having principal minors from graphs of type $E_{7}$ and $D_{6}$. Thus we get $\operatorname{det}(A)=2(2)-4=0$.
$\widetilde{E}_{8}$ We see that $\widetilde{E}_{8}$ comes from having principal minors from graphs of type $D_{8}$ and $A_{7}$. Thus we get $\operatorname{det}(A)=2(4)-8=0$.
$\widetilde{F}_{4}$ We see that $\widetilde{F}_{4}$ comes from having principal minors from graphs of type $F_{4}$ and $B_{3}$. Thus we get $\operatorname{det}(A)=2(1)-2=0$.
$\widetilde{G}_{2}$ We compute the determinant directly. We notice that for $\widetilde{G}_{2}$ we get $A=$ $\left(\begin{array}{ccc}2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2\end{array}\right)$. Therefore we get $\operatorname{det}(A)=8-2-6=0$.

We state the following lemma without proof. A proof may be found in [9].

Lemma 4.4.10. If $\Gamma$ is a connected Coxeter graph of positive type, then every proper subgraph is positive definite.

Finally, we prove the theorem.

Proof of theorem. (Outlined by [9]) We will actually prove something slightly stronger. We will prove not only that the figures in theorem (4.4.7), but that also those figure combined with the figures from lemma (4.4.9) are the only connected Coxeter graphs of positive type.

Suppose contrarily that there were another connected Coxeter graph $\Gamma$ of positive type that was not pictured in any of the 2 figures. Let's say that $\Gamma$ has $n$ vertices and $m$ is the maximum edge label. Note that no graph in lemma (4.4.9) can be a subgraph of $\Gamma$ by the previous lemma.
(1) All Coxeter graphs of dimension 1 or 2 are all clearly of positive type. We see this from graphs of type $A_{1}, I_{2}(m)$ and $\tilde{A}_{1}$. Therefore we must have $n>2$.
(2) Since $\tilde{A}_{1}$ cannot be a subgraph of $\Gamma$ we must have that $m<\infty$.
(3) Since $\tilde{A}_{n}$ for $n \geq 2$ cannot be a subgraph of $\Gamma, \Gamma$ contains no circuits.
(4) Suppose that $m=3$. Then $\Gamma$ must have a branch point somewhere as $\Gamma \neq A_{n}$.
(5) If $m=3$ then $\Gamma$ doesn't contain $\tilde{D}_{n}$ for $n>4$ so it must have a unique branch point.
(6) If $m=3$ then $\Gamma$ doesn't contain $\tilde{D}_{4}$ and thus exactly 3 edges must meet at the branch point.
(7) If $m=3$ and $a \leq b \leq c$ further vertices from the branch point, then since $\tilde{E}_{6}$ is not a subgraph of $\Gamma$ we have that $a=1$.
(8) If $m=3$ and $1 \leq b \leq c$ further vertices from the branch point, then since $\tilde{E}_{7}$ is not a subgraph of $\Gamma$ we have that $b \leq 2$.
(9) If $m=3$ and $1 \leq b \leq c$ further vertices from the branch point, then as $\Gamma \neq D_{n}$, $b \neq 1$. Therefore $b=2$.
(10) If $m=3$ and $1 \leq 2 \leq c$ further vertices from the branch point, then since $\tilde{E}_{8}$ is not a subgraph of $\Gamma$ then $c \leq 4$. If $m=3$ and $1 \leq 2 \leq c \leq 4$ further vertices from the branch point, then since $\Gamma \neq E_{6}, E_{7}, E_{8}$ then $c>4$ which is impossible and therefore $m \neq 3$.
(11) If $m \geq 4$, then since $\Gamma$ does not contain $\tilde{C}_{n}$, only one edge has a label $>3$.
(12) If $m \geq 4$, then $\Gamma$ does not contain $\tilde{B}_{n}$, so $\Gamma$ has no branch points.
(13) If $m=4$ then since $\Gamma \neq B_{n}$ we see that the two end vertices of $\Gamma$ have edges labelled 3.
(14) If $m=4$ then since $\Gamma$ does not contain $\tilde{F}_{4}$ we know that $n=4$.
(15) If $m=4$ then since $\Gamma \neq F_{4}$ we see that we can't have $m=4$. Therefore $m \geq 5$.
(16) If $m \geq 5$ since $\Gamma$ doesn't contain $\tilde{G}_{2}$ we must have that $m=5$.
(17) If $m=5$ we know that $\Gamma$ can't contain a nonpositive graph like

so the edge labelled 5 must be at an edge connected to an end vertex.
(18) If $m=5$ then $\Gamma$ can't contain the nonpositive graph

and therefore $n \leq 4$.
(19) If $m=5$, then $\Gamma$ must be either $H_{3}$ or $H_{4}$ which can't happen as they are already present in our figures.

Therefore no such $\Gamma$ exists and our claim is proved.

## Chapter 5

## Simple Lie Algebras

### 5.1 Connections with Crystallographic Root Systems ${ }^{[3],[9]}$

The final step left is to connect the world of Lie algebras and the world of Dynkin diagrams to show that we can in fact relate semisimple Lie algebras with their associated Dynkin diagrams. Earlier, we showed that if we let $L$ be a semisimple Lie algebra over the algebraically closed field $F$ with characteristic 0 we can use Cartan decomposition to get a maximal toral subalgebra $H$ of $L$. Not only that, but we can use the maximal toral subalgebra to get a set of roots of $L$ relative to $H$. This set was denoted $\Phi$ and is a subset of $H^{\star}$. We also saw in the creation of theorem 2.8.3 that the rational span of $\Phi$ in $H^{\star}$ is such that $\operatorname{dim}_{F} H^{\star}=n$. We then extended the base field from $\mathbb{Q}$ to $\mathbb{R}$, carried the dual of the Killing form over and obtained the Euclidean space $E$ spanned by $\Phi$. We then saw that $\Phi$ was a root system in $E$.

What is now left to do is to show that any two semisimple Lie algebras are isomorphic if and only if they have the same root system. We do this by first showing that simple Lie algebras have irreducible root systems. Then we break down semisimple Lie algebras into their simple components and then show that any two simple Lie algebras are isomorphic if they have the same irreducible root system.

We first show that $\Phi$ is irreducible when $L$ is simple.

Lemma 5.1.1. Let $L$ be a simple Lie algebra such that $H$ is its maximal toral subalgebra and $\Phi$ is the set of roots of $L$ relative to $H$. Then $\Phi$ is an irreducible root
system.
Proof. [8] We will suppose that $\Phi$ is not irreducible. Therefore there exist $\Phi_{1}$ and $\Phi_{2}$ such that $\Phi=\Phi_{1} \cup \Phi_{2}$ and the $\Phi_{i}$ are orthogonal. Let $\alpha_{1} \in \Phi_{1}$ and $\alpha_{2} \in \Phi_{2}$. We see that $\left(\alpha_{1}+\alpha_{2}, \alpha_{1}\right) \neq 0$ and $\left(\alpha_{1}+\alpha_{2}, \alpha_{2}\right) \neq 0$ and thus $\alpha_{1}+\alpha_{2}$ cannot be a root and therefore $\left[L_{\alpha_{1}}, L_{\alpha_{2}}\right]=0$. Therefore the subalgebra $K$ of $L$ generated by all $L_{\beta_{1}}$ $\left(\beta_{1} \in \Phi_{1}\right)$ is centeralised by all $L_{\beta_{2}}\left(\beta_{2} \in \Phi_{2}\right)$. This forces $K$ to be a proper subalgebra of $L$ as $Z(L)=0$. Not only that, but $K$ is normalised by $L_{\gamma}\left(\gamma \in \Phi_{1}\right)$ and thus by all $L_{\gamma}(\gamma \in \Phi)$ and thus by $L$ as $L$ is generated by the root spaces $L_{\gamma}$. Thus $K$ is a proper ideal of $L$, different from $\{0\}$ which contradicts the simplicity of $L$.

Next we show that each semisimple Lie algebra decomposed into simple ideals creates a decomposition of its root system into irreducible components.

Lemma 5.1.2. Let $L$ be a semisimple Lie algebra, with maximal toral subalgebra $H$ and root system $\Phi$. If $L=\bigoplus_{i=1}^{n} L_{i}$ is the decomposition of $L$ into simple ideals, then $H_{i}=H \cap L_{i}$ is a maximal toral subalgebra of $L_{i}$ and the corresponding irreducible root system $\Phi_{i}$ may be regarded as a subsystem of $\Phi$ such that $\Phi=\bigcup_{i=1}^{n} \Phi_{i}$ is the decomposition of $\Phi$ into its irreducible components.

Proof. (Described in [8]) From theorem 2.5.5 we know that $L$ can be decomposed into simple ideals $L=\bigoplus_{i=1}^{n} L_{i}$. Letting $H$ be the maximal toral subalgebra of $L$ we see it can similarily be decomposed into $H=\bigoplus_{i=1}^{n} H_{i}$ where each $H_{i}=H \cap L_{i}$. Each $H_{i}$ must be a toral algebra in $L_{i}$ and thus must also be maximal as any toral subalgebra of $L_{i}$ larger than $H_{i}$ would be toral in $L$, centralise all $H_{j}$ where $j \neq i$ and generate with them a toral subalgebra of $L$ larger than $H$ which is impossible as $H$ is maximal.

Now let $\Phi_{i}$ denote the root system of $L_{i}$ relative to $H_{i}$ in the Euclidean space $E_{i}$. If $\alpha_{i} \in \Phi_{i}$ we then can view $\alpha_{i}$ as a linear function on $H$ by letting $\alpha_{i}\left(H_{j}\right)=0$ for $j \neq i$. Thus $\alpha_{i}$ is a root of $L$ relative to $H$ such that $L_{\alpha_{i}} \subset L_{i}$. Conversely if $\alpha \in \Phi$ we see that $\left[H_{i}, L_{\alpha}\right] \neq 0$ for some $i$ otherwise $H$ would centralise $L_{\alpha}$. Thus $L_{\alpha} \subset L_{i}$ so $\left.\alpha\right|_{H_{i}}$ is a root of $L$ relative to $H_{i}$. Therefore $\Phi$ must be decomposed as stated in the lemma such that $E \cong \bigoplus_{i=1}^{n} E_{i}$.

Now we show that $L$ can be generated from its root spaces.
Lemma 5.1.3. Let $L, H$, and $\Phi$ be as before. Fix a simple system $\Delta$ of $\Phi$. Then $L$ is generated by the root spaces $L_{\alpha}, L_{-\alpha}(\alpha \in \Delta)$. Equivalently $L$ is generated by arbitrary
nonzero root vectors $e_{\alpha} \in L_{\alpha}, f_{\alpha} \in L_{-\alpha}(\alpha \in \Delta)$.
Proof. [8] Let $\beta \in \Pi$ be an arbitrary root. $\beta$ may be written as the linear combination of the simple roots. Therefore let $\beta=\sum_{i=1}^{s} \alpha_{i}$ where $\alpha_{i} \in \Delta$. We also know that [ $\left.L_{\gamma}, L_{\delta}\right]=L_{\gamma+\delta}$ whenever $\gamma, \delta, \gamma+\delta \in \Phi$. We next show that by induction on $s$ that $L_{\beta} \subseteq L$ such that $L$ is generated by all $L_{\alpha}$ where $\alpha \in \Delta$.

If we have that $s=1$ then we see that $\beta=c \alpha_{1}$ for $c \in F$ and thus since $\beta$ is a linear combination of $\alpha_{1}$ and $L_{\alpha_{1}}=L$ we see that $L_{\beta} \subseteq L_{\alpha_{1}}=L$. Now suppose that our hypothesis for $s>1$ and let $\beta=\sum_{i=1}^{s+1} \alpha_{i}$. Let $\gamma=\sum_{i=1}^{s} \alpha_{i}$ and thus $\beta=\alpha_{s+1}+\gamma$. Now we see that

$$
\begin{aligned}
L_{\beta} & =\{x \in L \mid[h, x]=\beta(h) x \quad \forall h \in H\} \\
& =\left\{x \in L \mid[h, x]=\left(\alpha_{s+1}+\gamma\right)(h) x \quad \forall h \in H\right\} \\
& =L_{\alpha_{s+1}+\gamma} \\
& =\left[L_{\alpha_{s+1}}, L_{\gamma}\right] \\
& \subseteq L
\end{aligned}
$$

and thus our claim is proved.
Similarly, if $\beta \in-\Pi$ then $L_{\beta}$ lies in the subalgebra of $L$ generated by all $L_{-\alpha}$ $(\alpha \in \Delta)$. But then $L=H \bigoplus_{\alpha \in \Phi} L_{\alpha}$ and $H=\sum_{\alpha \in \Phi}\left[L_{\alpha}, L_{-\alpha}\right]$ and thus the lemma follows.

Finally we relate root systems and simple Lie algebras to show that if two simple Lie algebras have the same root system then they are isomorphic.

Theorem 5.1.4. Let $L, L^{\prime}$ be two simple Lie algebras over $F$ with respective maximal toral subalgebras $H, H^{\prime}$ and root systems $\Phi, \Phi^{\prime}$. Suppose there is an isomorphism between $\Phi$ and $\Phi^{\prime}$ such that $\alpha \mapsto \alpha^{\prime}$ and inducing $\pi: H \rightarrow H^{\prime}$. Let $\Delta$ be a simple system for $\Phi$ and construct $\Delta^{\prime}=\left\{\alpha^{\prime} \mid \alpha \in \Delta\right\}$ such that $\Delta^{\prime}$ is a simple system for $\Phi^{\prime}$. Now for each $\alpha \in \Delta$ and $\alpha^{\prime} \in \Delta^{\prime}$ choose an arbitrary nonzero $e_{\alpha} \in L_{\alpha}$ and $e_{\alpha^{\prime}}^{\prime} \in L_{\alpha^{\prime}}$ (i.e. choose an arbitrary Lie isomorphism $\left.\pi_{\alpha}: L_{\alpha} \rightarrow L_{\alpha^{\prime}}^{\prime}\right)$.

Then there exists a unique isomorphism $\pi: L \rightarrow L^{\prime}$ extending $\pi: H \rightarrow H^{\prime}$ and extending all the $\pi_{\alpha}$ where $\alpha \in \Delta$.

### 5.2 Simple Lie Algebras Constructed ${ }^{[3],[8],[9]}$

In order to construct a simple Lie algebra for each of the Dynkin diagrams we use the approach as seen in [8]. We will generally first choose a lattice $L$ in $\mathbb{R}^{n}$ and define $\Phi$ to be the set of all vectors having one or two assigned lengths. We would then ensure that each $\langle\alpha, \beta\rangle$ are integers. From there it is easy to see that reflections with respect to vectors in $\Phi$ stabilize $L$ and therefore permute $\Phi$.

For everything in this section we denote $\varepsilon_{1}, \ldots, \varepsilon_{n}$ as the standard basis of $\mathbb{R}^{n}$. We also note that the subscript of each simple Lie algebra type is the rank of that Lie algebra.

### 5.2.1 $A_{n}$

We let $V$ be the hyperplane of $\mathbb{R}^{n+1}$ such that all coordinates in a particular vector in $V$ add up to 0 . This is the same construction we gave earlier. We let $\Phi$ be the set of all vectors such that their length squared is 2 and such that $\Phi \subseteq V \cap \mathbb{Z} \varepsilon_{1}+\ldots+\mathbb{Z} \varepsilon_{n+1}$. We can thus construct $\Phi$ to consist of the $n(n+1)$ vectors: $\varepsilon_{i}-\varepsilon_{j}$ such that $1 \leq i \neq$ $j \leq n+1$. We can take the simple system to be $\Delta=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ where $\alpha_{i}=\varepsilon_{i}-\varepsilon_{i+1}$. As $|\Phi|=n(n+1)$ and as we know that the rank of $A_{n}=n$ we see that the dimension of any Lie algebra with root system of type $A_{n}$ must be of dimension $n(n+2)$. By Ado-Iwasawa theorem we know there must be a linear Li algebra that is represented by $A_{n}$. This turns out to be the special linear algebra $\mathfrak{s l}(V)$ (or equivalently $\mathfrak{s l}(n+1, F)$ ) which is the subalgebra of $\mathfrak{g l}(V)$ in which every element has trace zero. We see that the Weyl group is $S_{n+1}$ which acts in the usual way by permuting the $\varepsilon_{i}$. Therefore $W$ has order $(n+1)$ !. We also note that the Cartan matrix is the $n \times n$ matrix

$$
\left(\begin{array}{ccccccc}
2 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & 0 & \ldots & -1 & 2
\end{array}\right)
$$

### 5.2.2 $B_{n}$

We let $V$ be equal to $\mathbb{R}^{n}$ and define $\Phi$ to be the set of all vectors whose squared length is 1 or 2 in the standard lattice. Therefore $\Phi$ consists of the short roots $\pm \varepsilon_{i}$ (of which there are $2 n$ ) and the long roots $\pm \varepsilon_{i} \pm \varepsilon_{j}$ for $i<j$ (of which there are $2 n(n-1))$. We therefore have a total of $2 n^{2}$ roots in $\Phi$. We can take the simple system to be $\Delta=\left\{\alpha_{1}, \ldots, \alpha_{n-1}, \alpha_{n}\right\}$ where $\alpha_{i}=\varepsilon_{i}-\varepsilon_{i+1}$ for $1 \leq i<n$ as before and we let $\alpha_{n}=\varepsilon_{n}$. As $|\Phi|=2 n^{2}$ and as we know that the rank of $B_{n}=n$ we see that the dimension of any Lie algebra with root system of type $B_{n}$ must be of dimension $2 n^{2}+n$. By Ado-Iwasawa theorem we know there must be a linear Lie algebra that is represented by $B_{n}$. This turns out to be the orthogonal linear algebra $\mathfrak{o}(2 n+1, F)$ which is the subalgebra of $\mathfrak{g l}(V)$ in which every element is skew-symmetric. We saw earlier that $W=S_{n} \rtimes(\mathbb{Z} / 2 \mathbb{Z})^{n}$ and thus $W$ has order $2^{n} n$ !. We can also deduce that the Cartan matrix is the $n \times n$ matrix

$$
\left(\begin{array}{ccccccc}
2 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 2 & -2 \\
0 & 0 & 0 & 0 & \ldots & -1 & 2
\end{array}\right)
$$

### 5.2.3 $C_{n}$

As we had mentioned earlier, $C_{n}$ is the dual root system of $B_{n}$. Thus we can just take the inverse of $B_{n}$ to get $C_{n}$. Thus we see that $C_{n}$ has long roots $\pm 2 \varepsilon_{i}$ and short roots $\pm \varepsilon_{i} \pm \varepsilon_{j}$ for $i<j$. The number of roots are the same and $\Delta$ consists of the same roots except that $\alpha_{n}=2 \varepsilon_{n}$. As $|\Phi|=2 n^{2}$ and as we know that the rank of $C_{n}=n$ we see that the dimension of any Lie algebra with root system of type $C_{n}$ must be of dimension $2 n^{2}+n$. By Ado-Iwasawa theorem we know there must be a linear Lie algebra that is represented by $C_{n}$. This turns out to be the symplectic linear algebra $\mathfrak{s p}(2 n, F)$ which is the subalgebra of $\mathfrak{g l}(V)$ in which every element is symplectic. $W$ is isomorphic to the same $W$ in $B_{n}$ and thus also has order $2^{n} n$ !. We can also deduce
that the Cartan matrix is the $n \times n$ matrix

$$
\left(\begin{array}{ccccccc}
2 & -1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & 0 & \ldots & -2 & 2
\end{array}\right)
$$

### 5.2.4 $D_{n}$

As in the case of $B_{n}$ we let $V=\mathbb{R}^{n}$. We let $\Phi$ be the set of all vectors of squared length 2 in the standard lattice. We thus see that $\Phi$ has all roots $\pm \varepsilon_{i} \pm \varepsilon_{j}$ for $1 \leq i<j \leq n$. We let the simple system $\Delta=\left\{\alpha_{1}, \ldots, \alpha_{n-1}, \alpha_{n}\right\}$ where $\alpha_{i}=\varepsilon_{i}-\varepsilon_{i+1}$ for $1 \leq i<n$ and $\alpha_{n}=\varepsilon_{n-1}+\varepsilon_{n}$. As $|\Phi|=2 n(n-1)$ and as we know that the rank of $D_{n}=n$ we see that the dimension of any Lie algebra with root system of type $D_{n}$ must be of dimension $2 n^{2}-n$. By Ado-Iwasawa theorem we know there must be a linear Lie algebra that is represented by $D_{n}$. This turns out to be the orthogonal linear algebra $\mathfrak{o}(2 n, F)$ just as for type $B_{n}$ except the dimension is even rather than odd. We saw earlier that $W=S_{n} \rtimes(\mathbb{Z} / 2 \mathbb{Z})^{n-1}$ and thus the order of $W$ is $2^{n-1} n$ !. We can also deduce that the Cartan matrix is the $n \times n$ matrix

$$
\left(\begin{array}{ccccccccc}
2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 0 & 2
\end{array}\right)
$$

### 5.2.5 $G_{2}$

We start with the smallest of the exceptional simple Lie algebras. For $G_{2}$ we let $V$ be the hyperplane in $\mathbb{R}^{3}$ whose vectors have coefficients adding up to 0 (just like for $A_{n}$. Unlike $A_{n}$ we let $\Phi$ to be the set of vectors of squared length 2 or 6 such that $\Phi$ is a
subset of the intersection of $V$ and the standard lattice. We see that the length ratio is 3 which will give us the triple edge we desire. $\Phi$ is the set of short roots $\pm\left(\varepsilon_{i}-\varepsilon_{j}\right)$ where $i<j$ and the long roots $\pm\left(2 \varepsilon_{i}-\varepsilon_{j}-\varepsilon_{k}\right)$ where $1 \leq i, j, k \leq 3$. We see that there are 12 roots altogether. We construct the simple system $\Delta$ to be the set of roots

$$
\begin{aligned}
& \alpha_{1}=\varepsilon_{1}-\varepsilon_{2} \\
& \alpha_{2}=-2 \varepsilon_{1}+\varepsilon_{2}+\varepsilon_{3} .
\end{aligned}
$$

As our rank is 2 we see that $G_{2}$ has dimension 14 and we note that the order of the Weyl group it generates is 12 . The Cartan matrix is the $2 \times 2$ matrix

$$
\left(\begin{array}{cc}
2 & -1 \\
-3 & 2
\end{array}\right)
$$

### 5.2.6 $F_{4}$

We next let $V=\mathbb{R}^{4}$. If we let $L^{s}$ be the standard lattice we can let $L:=L^{s}+$ $\mathbb{Z}\left(\frac{1}{2} \sum_{i=1}^{4} \varepsilon_{i}\right)$. It's easy to see that $L$ is also a lattice and thus we can define $\Phi$ to be the set of all vectors in $L$ of squared length 1 or 2 . Therefore $\Phi$ has long roots of the form $\pm \varepsilon_{i} \pm \varepsilon_{j}$ for $i<j$ and short roots $\pm \varepsilon_{i}$ and $\frac{1}{2}\left( \pm \varepsilon_{1} \pm \varepsilon_{2} \pm \varepsilon_{3} \pm \varepsilon_{4}\right)$. There are 24 long and 24 short roots giving us a total of 48 roots. To create a simple system we can let $\Delta$ bet he set consisting of the roots

$$
\begin{aligned}
\alpha_{1} & =\varepsilon_{2}-\varepsilon_{3} \\
\alpha_{2} & =\varepsilon_{3}-\varepsilon_{4} \\
\alpha_{3} & =\varepsilon_{4} \\
\alpha_{4} & =\frac{1}{2}\left(\varepsilon_{1}-\varepsilon_{2}-\varepsilon_{3}-\varepsilon_{4}\right) .
\end{aligned}
$$

As our rank is 4 we see that $F_{4}$ has dimension $4+48=52$ and we note that the order of the Weyl group it generates is $2^{7} \cdot 3^{2}$. From here we can construct the $4 \times 4$ Cartan matrix

$$
\left(\begin{array}{cccc}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{array}\right)
$$

### 5.2.7 $\quad E_{8}$

To construct $E_{7}$ and $E_{6}$ we must first look at $E_{8}$. We let $V=\mathbb{R}^{8}$ and we let $L:=$ $\sum_{i=1}^{8} c_{i} \varepsilon_{i}+\mathbb{Z}\left(\frac{1}{2} \sum_{i=1}^{8}\right)$ such that $c_{i} \in \mathbb{Z}$ and $\frac{1}{2} \sum c_{i} \in \mathbb{Z}$. We let $\Phi$ be the set of all vectors of length 2 in $L$. Therefore $\Phi$ has the roots $\pm \varepsilon_{i} \pm \varepsilon_{j}$ for $i<j$ and the roots $\frac{1}{2} \sum_{i=1}^{8} \pm \varepsilon_{i}$ (where there are an even number of + signs). We see that there must be 240 roots. We create a simple system $\Delta$ consisting of the following roots

$$
\begin{aligned}
\alpha_{1} & =\frac{1}{2}\left(\varepsilon_{1}-\varepsilon_{2}-\varepsilon_{3}-\varepsilon_{4}-\varepsilon_{5}-\varepsilon_{6}-\varepsilon_{7}+\varepsilon_{8}\right) \\
\alpha_{2} & =\varepsilon_{1}+\varepsilon_{2} \\
\alpha_{i} & =\varepsilon_{i-1}-\varepsilon_{i-2} \quad(3 \leq i \leq 8) .
\end{aligned}
$$

As our rank is 8 we see that $E_{8}$ has dimension $8+240=248$ and we note that the order of the Weyl group it generates is $2^{14} \cdot 3^{5} \cdot 5^{2} \cdot 7$. We thus construct the Cartan matrix

$$
\left(\begin{array}{cccccccc}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{array}\right)
$$

### 5.2.8 $\quad E_{7}$

To construct $E_{7}$ we let $L$ as in $E_{8}$, but we restrict $V$ to our first 7 roots (thus $V \subseteq$ $\mathbb{R}^{8} \cap \operatorname{span}\left\{\alpha_{i} \mid 1 \leq i \leq 7\right\}$ where $\alpha_{i}$ are the simple roots of $\left.E_{8}\right)$. Therefore we see that $\Phi$ is the set of all roots in $E_{8}$ that lie in $V$. These roots are $\pm \varepsilon_{i} \pm \varepsilon_{j}$ for $1 \leq i<j \leq 6$, $\pm\left(\varepsilon_{7}-\varepsilon_{8}\right)$ and $\frac{1}{2}\left(\varepsilon_{7}-\varepsilon_{8}+\sum_{i=1}^{6} \pm \varepsilon_{i}\right)$ (where there are an odd number of + signs). We see that $E_{7}$ has 126 roots. Our simple system $\Delta$ are the first 7 simple roots of $E_{8}$ as were used in the construction of $V$. As our rank is 7 we see that $E_{7}$ has dimension $7+126=133$ and we note that the order of the Weyl group it generates is $2^{10} \cdot 3^{4} \cdot 5 \cdot 7$.

We construct our Cartan matrix as

$$
\left(\begin{array}{ccccccc}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{array}\right)
$$

### 5.2.9 $\quad E_{6}$

For $E_{6}$ we let $L$ be as in $E_{8}$ again, but we restrict $V$ to our first 6 roots (thus $V \subseteq$ $\mathbb{R}^{8} \cap \operatorname{span}\left\{\alpha_{i} \mid 1 \leq i \leq 6\right\}$ where $\alpha_{i}$ are the simple roots of $\left.E_{8}\right)$. Therefore we see that $\Phi$ is the set of all roots in $E_{8}$ that lie in $V$. These roots are $\pm \varepsilon_{i} \pm \varepsilon_{j}$ for $1 \leq i<j \leq 5$ and $\frac{1}{2}\left(\varepsilon_{8}-\varepsilon_{7}-\varepsilon_{6}+\sum_{i=1}^{5} \pm \varepsilon_{i}\right)$ (where there are an odd number of + signs). We thus have 72 roots. Our simple system $\Delta$ are the first 6 simple roots of $E_{8}$ as were used in the construction of $V$. As our rank is 6 we see that $E_{6}$ has dimension $6+72=78$ and we note that the order of the Weyl group it generates is $2^{7} \cdot 3^{4} \cdot 5$. We construct our Cartan matrix as

$$
\left(\begin{array}{cccccc}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{array}\right)
$$

## Chapter 6

## Conclusion

We have thus classified all semisimple Lie algebras over an algebraically closed field of characteristic 0 . Although it might seem like we know all there is to know, there are still vast areas to explore within Lie algebras, Coxeter groups, and root systems in general.

For Lie algebras we could turn to finding the simple Lie algebras of algebraically closed fields of positive characteristic. All simple Lie algebras for characteristic $p \geq 5$ have been found by Block-Wilson-Premet-Stride. The classification of all simple Lie algebras for characteristic $p=2,3$ still remains an open problem.

Alternatively, one can continue down the road of Coxeter groups. Coxeter groups have a rich combinatorial structure and are used to find Catalan numbers, Fuss-Catalan numbers, and Fuss-Narayana to name a few. Recently noncrossing partition graphs have been employed to help find these numbers using the appropriate zeta polynomials.

Finally, we can continue down the road of root systems and try to see their geometric structures. By exploring the geometric structures of the root systems we can get a better understanding of how these structures work and how they might be applied to other applications.

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