Math 3094 Humphreys Sec 1.3 Notes

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Fall 2018

Proof of Theorem in Section 1.3 of Humphreys

Let Π be a positive system of a root system Φ. Suppose D is a minimal subset of Φ subject to the requirement that

$$\text{each root in } \Pi \text{ is a nonnegative linear combination of } D. \tag{1}$$

Prove that

$$\langle \alpha, \beta \rangle \leq 0 \text{ for all pairs } \alpha \neq \beta \in D.$$

How to do this problem: This inequality statement is labeled (1) on [Hum90, page 8]. The inequality is proven on [Hum90, page 9]. You can also read [Der14, page 34].

Proof. Suppose $\langle \beta, \alpha \rangle > 0$ for some $\alpha \neq \beta \in D$. Then the formula for a reflection gives

$$s_\alpha(\beta) = \beta - c\alpha, \quad \text{with } c = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} > 0.$$

To see this, recall that $\langle \alpha, \alpha \rangle > 0$ because $\langle , \rangle$ is an inner product (*i.e.*, positive definite symmetric bilinear form) on $\mathbb{R}^n$. Since $s_\alpha(\beta) \in \Phi = \Pi \cup -\Pi$, either $s_\alpha(\beta) \in \Pi$ or $-s_\alpha(\beta) \in \Pi$. The argument that the former case is impossible is done in [Hum90, Section 1.3], and we will now show that the latter case is also impossible.

Suppose $-s_\alpha(\beta) \in \Pi$. Then

$$\begin{aligned} -s_\alpha(\beta) &= \sum_{\gamma \in \Delta} c_\gamma \gamma \text{ with } c_\gamma \geq 0 \text{ by definition of } D \\ &= c_\alpha \alpha + \sum_{\gamma \in \Delta, \gamma \neq \alpha} c_\gamma \gamma. \end{aligned}$$

Hence

$$-s_\alpha(\beta) = -\beta + c\alpha = c_\alpha \alpha + \sum_{\gamma \in \Delta, \gamma \neq \alpha} c_\gamma \gamma. \tag{2}$$

In case $c_\alpha < c$, equation (2) implies

$$(c - c_\alpha)\alpha = \beta + \sum_{\gamma \in \Delta, \gamma \neq \alpha} c_\gamma \gamma,$$

which is a non-negative linear combination of $D \backslash \{\alpha\}$. Since $0 < c - c_\alpha$, this allows us to discard α from the subset D and still have a subset satisfying the requirement (1), contradicting the minimality of D.

In case $c < c_\alpha$, equation (2) gives

$$0 = (c_\alpha - c)\alpha + \beta + \sum_{\gamma \in \Delta, \gamma \neq \alpha} c_\gamma \gamma,$$

which is a non-negative linear combination of D. But all these roots are positive and the coefficient for β has a positive coefficient (at least 1), so the right hand side cannot equal to 0 (by definition of total ordering).

We've shown that $-s_\alpha(\beta) \notin \Pi$. Since $s_\alpha(\beta) \notin \Pi$ as shown in [Hum90, Section 1.3], we have $\alpha(\beta) \notin \Phi$, which is a contradiction. $\square$

References

[Der14] Aram Dermenjian. *Crystallographic Root Systems*, 2014. `https://egunawan.github.io/coxeter/text/dermenjian_survey_crystallographic_root_systems.pdf`.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.