MATH 3094 WEEK 11 PROBLEM SET BY YOURNAME

YOUR PREFERRED FIRST AND LAST NAME

Credit: Write down everyone who helped you, including classmates who contributed to your thought process (either through sharing insights or through being a sounding board). Write down Judson's textbook and other written sources you used as well.

Required
— Emily

Please remove this instruction section when you are done.
— Emily

Instruction.

Complete this problem set by LaTeX. Please send me an invite via Overleaf by entering my email address.

Note: If you are not sure how to do something, please post on Piazza or come to office hour.

1. Humphreys Sec 2.9, page 39-40 (Due Tuesday; May be handwritten)

The definitions (crystallographic and $\hat{L}$) are listed on page 39-40.

(1) Show that the action of W on $\hat{L}(\Phi)$ is stable if Φ is a crystallographic root system.

Proof. Suppose Φ is a crystallographic root system. Since W is generated by reflections σ_γ, $\gamma \in \Phi$, It is enough to show that

$$\sigma_\gamma\left(\hat{L}(\Phi)\right) \subseteq \hat{L}(\Phi) \text{ for all } \gamma \in \Phi.$$

Let $\lambda \in \hat{L}(\Phi)$ and let $\beta \in \Phi$. Then $\langle \lambda, \alpha^V \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$.

Fill in the rest of the proof. Prove that $\langle \sigma_\beta(\lambda), \alpha^V \rangle$ is also an integer for all $\alpha \in \Phi$.
— Emily

$\square$

2. Humphreys Sec 1.3, page 7-9 (Due Tuesday; May be handwritten)

Write down the statement of Theorem part (b) from Sec 1.3, page 8. Then prove part (b) by first carefully reading the proof in the book and then explaining it in more details. You can skip the last paragraph (which you've already done in a previous handwritten HW).

3. Humphreys Sec 1.4 (Due Tuesday; May be handwritten)

Write down the proposition in Sec 1.4 and write the proofs (include a little bit more details than what is given in the textbook). This part can be handwritten if you prefer.

4. Humphreys Sec 1.5 Exercise 2 (LaTeX only)

a. Type the Corollary from Sec 1.5:
b. Exercise 2 of Sec 1.5. Let Δ be a simple system. Then no proper subset of Δ can generate W.
   Hints:
   - Step 1: Otherwise, there is a simple reflection $\alpha \in \Delta$ which is not needed as a generator of W.
   - Step 2: Use the Corollary from Sec 1.5 to justify that there is $w \in W$ such that $w(-\alpha) \in \Delta$
   - Step 3: Apply Proposition in Sec 1.4 to say that every simple reflection σ_β sends all negative roots to negative roots (except for $-\beta$).
   - Step 4: Conclude that w cannot send $-\alpha$ to a positive root, in particular, w cannot send $-\alpha$ to a simple reflection in Δ, which is a contradiction.

5. Humphreys Sec 1.5 Exercise 3 (LaTeX only)

a. (May be handwritten) Write down the Theorem in Sec 1.5. Write down just Step (1) of the proof. Include a little bit more details than what is given in the textbook. This part can be handwritten if you prefer.
b. Exercise 3. Fix a simple system Δ and corresponding positive system Π in a root system Φ. Suppose $\beta \in \Pi \setminus \Delta$. Prove that $\mathrm{ht}(\beta) > 1$.
   Hints: Use similar strategy as Step (1) of the proof of Theorem in Sec 1.5:
   - Consider a set $P = \{\beta \in \Phi | \beta \in \Pi, \beta \notin \Delta \text{ and } \mathrm{ht}(\beta) \leq 1\}$.
   - For the sake of contradiction, suppose P is nonempty.

- As in Step (1), choose an element $\gamma \in P$ of smallest possible height.
- Since $\gamma \in Pi$, As in Step (1), write γ as a linear combination of the simple reflections of Δ with non-negative coefficients. Pick out $\alpha' \in \Delta$ where the positive coefficients $c_{\alpha'}$
- As in Step (1), use Proposition in Sec 1.4 to claim that $\sigma_{\alpha'}(\gamma) \in \Pi$ for all these α' because $\gamma \neq \alpha$ (why is $\gamma \neq \alpha'$?)
- As in step (1), consider what $\sigma_{\alpha'}(\gamma)$ is using the formula at the first page of Chapter 1.
- As in step (1), explain why you get a contradiction.

6. Miscellaneous

(1) Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.

(2) Approximately how much time did you spend on this homework?