Bonus questions

1. (Favorite theorems) What three theorems or concepts did you most enjoy from the course, and why? Choose one theorem/proposition/lemma of moderate difficulty and reconstruct its proof .
2. (Reflective question) Reflect on your overall experience in this class by describing an interesting idea that you learned, why it was interesting.
3. Write a joke.

Selected from Judson 5.3 permutation groups

http://abstract.ups.edu/aata/section-permutation-definitions.html

(a) Exercise 22. Prove Theorem 5.15: If a permutation σ can be expressed as the product of an even number of transpositions, then any other product of transpositions equaling σ must also contain an even number of transpositions. Similarly, if σ can be expressed as the product of an odd number of transpositions, then any other product of transpositions equaling σ must also contain an odd number of transpositions.
You may use Lemma 5.14.

(b) Prove Proposition 5.17: Let $n > 1$. The number of even permutations in S_n is equal to the number of odd permutations; hence, the order of A_n is $n!/2$. (Note: we proved that λ_σ is surjective in class).

From Computation HW 2 Exercises

Judson Chapter 5 Exercises http://abstract.ups.edu/aata/exercises-permute.html

i) Compute Exercise 2f,g,i,j,n,o,p.

ii) Exercise 3b,d.

iii) Exercise 4. Find the inverse of $(a_1, a_2, a_3, \ldots, a_n)$.

iv) **Definition:** The *order* of a permutation is the smallest positive integer m such that $a^m = id$. For example, the order of $(1, 2)(3, 4)$ is 2, and so is the order of $(1, 2)$. The order of $(1, 2, 3, 4)$ is 4, and the order of $(1, 2)(3, 4)(5, 6, 7)$ is 6. The order of the identity permutation is 1.

Exercise 8. Find a permutation in A_{10} with order 15. Write this permutation as a product of transpositions.

v) Exercise 9. *Can you* find a permutation in A_8 with order 26?

vi) Exercise 13. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_m \in S_n$ be the product of disjoint cycles. What is the order of σ?

Selected from Week 2 Problem Set

Judson's Exercises Chapter 5: http://abstract.ups.edu/aata/exercises-permute.html

i. Exercise 21. Let $\sigma \in S_n$ be not a cycle. Prove that σ can be written as the product of at most $n - 2$ transpositions.

ii. Prove Exercise 23. If σ is a cycle of odd length, prove that σ^2 is also a cycle.

iii. A variant of Exercise 23. If σ is a cycle of even length (longer than 2), describe σ^2.

iv. Lemma for Exercise 26.

a. Write the transposition $(a\,b)$ as a finite product of

$$(12), (13), (14), \ldots, (1n).$$

Warm-up: First try writing a product for $a = 2$, $b = 5$.

b. Write the transposition $(a\,b)$ as a finite product of

$$(12), (23), (34), \ldots, (n-1, n).$$

Warm-up: First try writing a product for $a = 2$, $b = 5$.

c. Write the transposition $(a\,b)$ as a finite product of the two cycles

$$(12) \text{ and } (123\ldots n).$$

Warm-up: First try writing a product for $a = 2$, $b = 5$.

v. Exercise 26.

a. Prove that any permutation in S_n can be written as a finite product of $(12), (13), (14), \ldots, (1n)$.

b. Prove that any permutation in S_n can be written as a finite product of $(12), (23), (34), \ldots, (n-1, n)$.

c. Prove that any permutation in S_n can be written as a finite product of the two cycles (12) and $(123\ldots n)$.

vi. Exercise 30. Let $\tau = (1, 2, 3, \ldots, k)$.

a. Prove that if σ is any permutation, then

$$\sigma\tau\sigma^{-1} = (\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(k))$$

b. Let $\mu = (b_1, b_2, \ldots, b_k)$ be a cycle of length k. Find a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

Selected from Judson 3.2 groups definitions and examples

http://abstract.ups.edu/aata/section-groups-define.html

(a) Show that $GL_2(\mathbb{R})$ (the set of 2×2 invertible matrices) is a group. (See proof in Example 3.14.)

Selected from Judson 3.2 Basic Properties of Groups

http://abstract.ups.edu/aata/section-groups-define.html#groups-subsection-basic-properties

(a) Prove Proposition 3.17: The identity element in a group G is unique; that is, there exists only one element $e \in G$ such that $eg = ge = g$ for all $g \in G$.

(b) Prove Proposition 3.18: If g is any element in a group G, then the inverse of g, denoted by g^{-1}, is unique.

Selected from Judson 3.3 Subgroups

http://abstract.ups.edu/aata/section-subgroups.html

(a) Prove that the set $SL_2(\mathbb{R})$ (of 2×2 matrices with determinant 1) is a subgroup of $GL_2(\mathbb{R})$. See Example 3.26.

(b) Prove Proposition 3.31: Let H be a subset of G. Then H is a subgroup of G if and only if $H \neq \emptyset$, and whenever $g, h \in H$ then $gh^{-1} \in H$.

Selected from Week Four HW, from Judson Ch 3 Exercises

http://abstract.ups.edu/aata/exercises-groups.html

i. Exercise 2a-d. Which tables form a group?

ii. Exercise 47. Prove or disprove: If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G. If true, you can prove using the subgroup proposition. If false, provide a counterexample.

iii. Variation of Exercise 47. Prove or disprove: If H and K are subgroups of a group G, then $H \cap K$ is a subgroup of G. If true, you can prove using the subgroup proposition. If false, provide a counterexample.

iv. Exercise 49. Let a and b be elements of a group G. If $a^4b = ba$ and $a^3 = e$, prove that $ab = ba$. Hint: replace a^4b with $(a^3)ab$.

Selected from Week Four Problem Set, from Judson Chapter 3

Judson Chapter 3 Exercises: http://abstract.ups.edu/aata/exercises-groups.html

i. Exercise 7. Let $S = \mathbb{R}\setminus\{-1\}$. Define a binary operation on S by $a \star b = a + b + ab$. Prove that $(S, \star)$ is an Abelian group. That is, after checking that the set S is closed under $\star$, you only need to check that (i) $\star$ is associative, (ii) S contains an identity, (iii) every element has an inverse, and (iv) every pair of elements commute.

ii. Exercise 28. Prove the first or second half of Proposition 3.21: Let G be a group and a and b be any two elements in G. Then the equations $ax = b$ and $xa = b$ have unique solutions in G.

iii. Exercise 32. Show that if G is a finite group of even order, then there is an $a \in G$ such that a is not the identity and $a^2 = e$.

iv. Exercise 34. Find all the subgroups of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Use this information to show that $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is not the same group as $\mathbb{Z}\setminus 9\mathbb{Z}$. Hint: See Example 3.28 in http://abstract.ups.edu/aata/section-subgroups.html, which explains why $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not the same group as $\mathbb{Z}/4\mathbb{Z}$.

v. Exercise 48. Let G be a group and g a fixed element of G. Show that

$$Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$$

is a subgroup of G.

(You can use one of the subgroup theorems in http://abstract.ups.edu/aata/section-subgroups.html#groups-subsection-subgroup-theorems).

Note: $Z(G)$ is called the *center* of G.

vi. Exercise 51. If $xy = x^{-1}y^{-1}$ for all x and y in G, prove that G must be abelian.
Hint: Since $x = x^{-1}$ for all $x \in G$, you have $ab = bbab = baabab$.

vii. Exercise 53. Let G be a group and H a subgroup of G. Show that

$$C(H) = \{g \in G : gh = hg \text{ for all } h \in G\}$$

is a subgroup of G.

(You can use one of the subgroup theorems in http://abstract.ups.edu/aata/section-subgroups.html#groups-subsection-subgroup-theorems).

Note: $C(G)$ is called the *centralizer* of G.

viii. Exercise 54. Let G be a group and let H be a subgroup of G. If $g \in G$, define

$$gHg^{-1} := \{ghg\}.$$

Show that gHg^{-1} is a subgroup of G. (You can a the subgroup theorems in http://abstract.ups.edu/aata/section-subgroups.html#groups-subsection-subgroup-theorems). Note: This subgroup is called the centralizer of H in G.

Selected from Judson 4.1 Cyclic subgroups

http://abstract.ups.edu/aata/section-cyclic-subgroups.html

(a) Example 4.5. Consider the cyclic subgroup $\mathbb{Z}_6$. What are the subgroups of $\mathbb{Z}_6$? Describe the subgroup generated by 2. Describe the subgroup generated by 3. Describe the subgroup generated by 3 and 4.

(b) Example 4.15. Consider the cyclic subgroup $\mathbb{Z}_{16}$. What are the subgroups of $\mathbb{Z}_{16}$? Describe the subgroup generated by 6. Describe the subgroup generated by 3.

(c) What are the subgroups of $\mathbb{Z}$? See Corollary 4.11.

Selected from Judson 11.1 Group homomorphisms

http://abstract.ups.edu/aata/section-group-homomorphisms.html

I Prove Proposition 11.4 (except for the second half of part 4): Let $f : G_1 \to G_2$ be a homomorphism of groups. Then

1. If e_1 is the identity of G_1, then $f(e_1)$ is the identity of G_2.
2. For any element $g \in G_1$, $f(g^{-1}) = [f(g)]^{-1}$.
3. If H_1 is a subgroup of G_1, then $f(H_1)$ is a subgroup of G_2.
4. If H_2 is a subgroup of G_2, then $f^{-1}(H_2) = \{g \in G_1 : f(g) \in H_2\}$ is a subgroup of G_1.

II Let $f : G_1 \to G_2$ be a homomorphism of groups. Prove that the kernel of f is a subgroup of G_1. (The proof of this is available right below Proposition 11.4).

III Example 11.7 (see class notes). Consider the group homomorphism $f : (\mathbb{R}.+) \to (\mathbb{C}^*, \times)$ defined by

$$f(\theta) = \cos\theta + i\sin\theta.$$

What is the kernel of f?
Give a bijective group homomorphism from the kernel of f to $(\mathbb{Z}, +)$. Prove that this map is a group homomorphism.

IV Example 11.6. What is the kernel of the group homomorphism $f : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by $f(A) = \det(A)$?

V Example 11.8 (see class notes). Determine all possible homomorphisms f from $\mathbb{Z}_7$ to $\mathbb{Z}_{12}$. Determine all possible homomorphisms f from $\mathbb{Z}_{17}$ to $\mathbb{Z}_{12}$.

VI Example 11.9 (see class notes). Let G be a group and let g be some element in G. Consider the group homomorphism from $\mathbb{Z}$ to G given by $f(n) = g^n$.

(a) If the order of g is infinite, what is the kernel of f? justify.
(b) If the order of g is finite, say k, what is the kernel of f? justify.

Selected from from Week Seven HW, from Judson 11.3

Sec 11.3 Exercises http://abstract.ups.edu/aata/exercises-homomorph.html

i. Exercise 1. Prove that $\det(AB) = \det(A) \quad \det(B)$ for $A, B \in GL_2(\mathbb{R})$.
(Note: This shows that the determinant is a group homomorphism from $GL_2(\mathbb{R})$ (with matrix multiplication as the group operation) to $\mathbb{R}^*$, the nonzero real numbers, with multiplication as the group operation.)

ii. Exercise 2. Consider the four maps ϕ between groups defined on Exercise 2 parts (a)-(e).
   - For each map, determine whether it is a homomorphism or not a homomorphism.
   - For each map which is a homomorphism, what is the kernel?
   - For each map which is not a homomorphism, explain why (give an example).

iii. Exercise 4. Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ be the map given by $\phi(n) = 7n$ for $n \in \mathbb{Z}$. Find the kernel and the image of ϕ.

iv. (Not from Judson) Prove or disprove the following: if G is a cyclic group and H is a subgroup of G, then H is also a cyclic group.
   Hint: You can study and paraphrase Section 2 of Conrad's subgroups of cyclic groups notes `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/cyclicgp.pdf`. Warning: many examples shown are for groups with multiplication as the group operation, unlike our problems.

v. We proved Exercise 18 in class: Let $f : G \to H$ be a group homomorphism. Show that f is one-to-one if and only if the kernel of f is 1_G.

vi. Exercise 19 part i and ii. Given a homomorphism $\phi : G \to H$ define a relation $\sim$ on G by $a \sim b$ if $\phi(a) = \phi(b)$ for $a, b \in G$. Show that this relation is an equivalence relation. Describe the equivalence classes.

vii. (Not from Judson) Let

$$S_n^B := \{\text{bijections } w : [\pm n] \to [\pm n] \text{ where } w(-a) = -w(a)\}.$$

   Prove that this set S_n^B is closed under composition (using just the definition of the set).

viii. Answer the following using the formula of a reflection associated to a vector, on the first page of Humphreys.
   - Prove that σ_α is an orthogonal transformation. Use the bilinearity of $<,>$.
   - Prove that, if $\beta \in H_\alpha$, then σ_α sends β to itself.
   - Prove or disprove that σ_α^2 is the identity map.

Bjorner Brenti 1.1

(a) You will get an example of a Coxeter system, for example, see page 2. Go from the Coxeter matrix to Coxeter graph and (generators plus relations) or in other directions.

Bjorner Brenti 1.2

(a) Example 1.2.1. Consider Coxeter graph with n isolated vertices (no edges). Describe the Coxeter matrix. What are the relations of the Coxeter group corresponding to this graph? Describe the Coxeter group.

(b) Example 1.2.2. Describe the Coxeter group associated to the Coxeter graph which is the complete graph on n vertices with all edges labeled by the infinity symbol.

(c) Example 1.2.3. Consider Coxeter graph which is a path with $n - 1$ vertices. Describe the Coxeter matrix. What are the relations of the Coxeter group corresponding to this graph? Describe the Coxeter group.

(d) Example 1.2.4. Consider Coxeter graphs with n vertices of type B_n and D_n. Describe the Coxeter matrices. What are the relations of the Coxeter groups corresponding to these graphs? Describe the Coxeter group.

(e) Example 1.2.5. Consider Coxeter graph which is a cycle of n vertices. Describe the Coxeter matrix. What are the relations of the Coxeter group corresponding to this graph?

(f) Example 1.2.7. Consider Coxeter graph on 2 vertices with one edge labeled by m. Describe the Coxeter matrix. What are the relations of the Coxeter group corresponding to this graph? Describe the Coxeter group.