Math3094 Final Exam

replace with your name

Instruction

remove this instruction section when you are done.

— Emily

- Deadline for sharing Overleaf file: Friday, December 14 (noon).

  Note 1: If you wish to continue making changes to your Overleaf file (until Sunday, December 16 at noon), you may let me know, and I will wait to download your submission until then.

  Note 2: You may share with me your Overleaf file prior to Friday if you have questions about what you've done so far.

- You may and should use [Hum90], [BB05], class notes (including a copy of a classmate's class notes in case you missed some classes), previous homework and problem sets.
- You may use all theorems stated in class (except for problems where you are asked to prove something 'from scratch').
- You may use Sage and other technology to verify your permutation multiplication. Note that the order of composition of Sage (left to right) is opposite our convention (right to left).
- During this take-home exam, you may *not* consult other resources (other textbooks, the internet, etc). You are not to discuss the exam with individuals other than the instructor until after the deadline.
- Two problems (pages) will be dropped, so please complete at least four out of six problems (pages). I will read your entire submission.
- Note that the last item on each of problems 2-5 is the most challenging. Even if you do not complete this last item, you should still complete the earlier (computational) items for partial credit.

1. Let W be a finite reflection group and fix a simple system Δ. Given $x \in W$, define the *length* $\ell(x)$ of x as in [Hum90, Sec 1.6, pg 12] (also stated in [BB05, Sec 1.4, pg 15]).

   Define

$$\begin{aligned} &d : W \times W \to [0, \infty) \ \text{ by} \\ &d(x, y) := \ell(xy^{-1}) \end{aligned}$$

   Show that d is a metric on W, that is, show that, for all $x, y, z \in W$, we have

   a. $\ell(xy^{-1}) \geq 0$
   b. $\ell(xy^{-1}) = 0$ if and only if x=y
   c. $\ell(xy^{-1}) = \ell(yx^{-1})$
   d. $\ell(xz^{-1}) \leq \ell(xy^{-1}) + \ell(yz^{-1})$ (triangle inequality)

   Proof. Click here for a hint: □

2. Consider the symmetric group S_n. For $i = 1, 2, \ldots, n-1$, set

$$s_i := (i, i+1) = \begin{pmatrix} 1 & 2 & \ldots & i & i+1 & \ldots & n \\ 1 & 2 & \ldots & i+1 & i & \ldots & n \end{pmatrix}.$$

Given $x = \begin{pmatrix} 1 & 2 & \ldots & n \\ x(1) & x(2) & \ldots & x(n) \end{pmatrix} \in S_n$, define

$$\begin{aligned} \mathrm{INV}(x) &:= \{ \text{ pair } (i,j) : i < j, x(i) > x(j)\} \quad \text{and let} \\ \mathrm{inv}(x) &:= \text{card } \mathrm{INV}(x). \end{aligned}$$

a. Choose a specific example of a permutation $x \in S_5$ where $\mathrm{inv}(x) > 0$. Write x in either window or 2-line notation. Compute $x\, s_2$ and write it in either window or 2-line notation. Write down $\mathrm{INV}(x)$ and $\mathrm{INV}(x\, s_2)$ and compare these two sets.

b. Consider an arbitrary permutation $x = \begin{pmatrix} 1 & 2 & \ldots & n \\ x(1) & x(2) & \ldots & x(n) \end{pmatrix} \in S_n$. Write $x\, s_i$ in window notation or 2-line notation.

c. Prove the following from scratch (don't use other theorems).

   I. If $x(i) < x(i+1)$, then $\mathrm{inv}(xs_i) = \mathrm{inv}(x) + 1$.

   II. If $x(i) > x(i+1)$, then $\mathrm{inv}(xs_i) = \mathrm{inv}(x) - 1$.

3. Consider the symmetric group $W = S_n$. As in [Hum90, Sec 2.10, pg 41], fix the simple system $\Delta = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n\}$ which corresponds to the set of simple transpositions $S = \{s_1 = (1, 2), s_2 = (2, 3), \ldots, s_{n-1} = (n-1, n)\}$.

   Given $x \in S_n$, define the *length* $\ell(x)$ of x as in [Hum90, Sec 1.6, pg 12] (or [BB05, Sec 1.4, pg 15]).

   Show that $\ell(x) = \mathrm{inv}(x)$ for all $x \in S_n$, where $\mathrm{inv}(x)$ is defined in Problem 2.

   Proof. Hint 1: You may cite Problem 2 even if you omit it.

   Click here for Hint 2:

   $\square$

4. Keep the same setup as in Problem 3. The positive system corresponding to Δ is $\Pi := \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\}$ which corresponds to the set of transpositions $\{(i,j) \in S_n : 1 \leq i < j \leq n\}$. Note that $\operatorname{card} \Pi = n(n-1)/2$.

   Let w_0 denote the unique element which sends Π to $-\Pi$, see [Hum90, Sec 1.8, pg 15-16]. Since $\ell(w_0) = \operatorname{card}\{\text{positive roots sent to negative roots by } w_0\}$ (due to [Hum90, Corollary 1.7, pg 14]), we have $\ell(w_0) = \operatorname{card} \Pi = n(n-1)/2$.

   By Problem 3, we have $\operatorname{inv}(w_0) = \ell(w_0) = n(n-1)/2$. Since $\operatorname{card}\{(i,j) : 1 \leq i < j \leq n\} = n(n-1)/2$, we must have $\operatorname{INV}(w_0) = \{\text{ pair } (i,j) : 1 \leq i < j \leq n\}$. So $w_0(1) > w_0(2) > \cdots > w_0(n)$, and hence

$$w_0 := \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}.$$

   a. (Optional) If you haven't already done so in this exam, prove the following.
      If $x(i) < x(i+1)$, then $(i,i+1) \notin \operatorname{INV}(x)$ and $(i,i+1) \in \operatorname{INV}(xs_i)$.
      If $x(i) > x(i+1)$, then $(i,i+1) \in \operatorname{INV}(x)$ and $(i,i+1) \notin \operatorname{INV}(xs_i)$.

      Proof. $\square$

   b. Write down $w_0\,x\,s_i$ in either window or 2-line notation. Click here for hints:

   c. Prove the following.

      I. If $x(i) < x(i+1)$, then
         $(i,i+1) \in \operatorname{INV}(w_0\,x)$ and $(i,i+1) \notin \operatorname{INV}(w_0\,x\,s_i)$.

      II. If $x(i) > x(i+1)$, then
         $(i,i+1) \notin \operatorname{INV}(w_0\,x)$ and $(i,i+1) \in \operatorname{INV}(w_0\,x\,s_i)$.

      Proof. Click here for a hint: $\square$

5. Keep the same setup as Problem 4. Let

$$\begin{aligned}\mathrm{INV}^C(x) :&= \{(i,j) : 1 \le i < j \le n, (i,j) \notin \mathrm{INV}(x)\}\\ &= \{(i,j) : 1 \le i < j \le n\} \setminus \mathrm{INV}(x)\end{aligned}$$

denote the 'complement' of $\mathrm{INV}(x)$. Note that card $\mathrm{INV}^C(x) = \ell(w_0) - \mathrm{inv}(x) = n(n-1)/2 - \mathrm{inv}(x)$.

a. In window notation, let $x := [2143]$ (length 2) and $y := [3412]$ (length 4) in S_4. Compute $\mathrm{INV}(x)$, $\mathrm{INV}^C(x)$ and $\mathrm{INV}(y)$, and verify that $\mathrm{INV}^C(x) = \mathrm{INV}(y)$. (Note that $2 + 4 = 6 = 4(3)/2 = n(n-1)/2 = \ell(w_0)$. Use Figure 1 to guess a pattern.)

b. Let $x := [4123]$ (length 3). Compute $\mathrm{INV}(x)$ and $\mathrm{INV}^C(x)$. Find $y \in S_n$ (length 3) and such that $\mathrm{INV}^C(x) = \mathrm{INV}(y)$. (Use Figure 1 to guess a pattern.)

c. Let $x := [3241]$ (length 4). Compute $\mathrm{INV}(x)$ and $\mathrm{INV}^C(x)$. Find $y \in S_n$ (length 2) and such that $\mathrm{INV}^C(x) = \mathrm{INV}(y)$. (Use Figure 1 to guess a pattern.)

d. Show that if $x \in S_n$ then there exists $y \in S_n$ such that $\mathrm{INV}^C(x) = \mathrm{INV}(y)$.
Note: the set equality implies that, if $\ell(x) = r$ then $\ell(y) = \ell(w_0) - r$.
Click here for hints:

6. Recall that the Coxeter group of type H_3 can be generated by elements $\{a, b, c\}$ with relations $(ab)^5 = (bc)^3 = (ac)^2 = a^2 = b^2 = c^2 = 1$. See Figure 2.

   a. Label all the edges of the weak order of H_3, where u and v are connected by an edge labeled s if and only if $u = v\,s$ (if and only if $v = u\,s$). In Figure 2, I have started to label some edges.

      My suggestion would be to draw all the edges by hand then scan/photograph it. Add your finished figure into your Overleaf document. If you have trouble adding a figure, please send me your figure and I will add it to your Overleaf document.

      Note: This is not meant to take a lot of time. Let me know if you don't understand my examples and aren't sure how to write the labels quickly.

   b. Use your labeled figure to:
      i.) write a reduced expression of the longest element w_0 of H_3, which is of length 15 and located at the top of the graph.
      ii.) find a reduced expression that is a power of 5, that is, write $w_0 = w^5$ for some reduced expression of an element w of length 3.

   c. Consider the three elements labeled x, y, and z appearing in Figure 2. Exactly one of them is the product of the other two. Determine the corret identity (among the six possible identities, i.e. $xy = z$, $yx = z$, $xz = y$, etc.).

References

[BB05] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

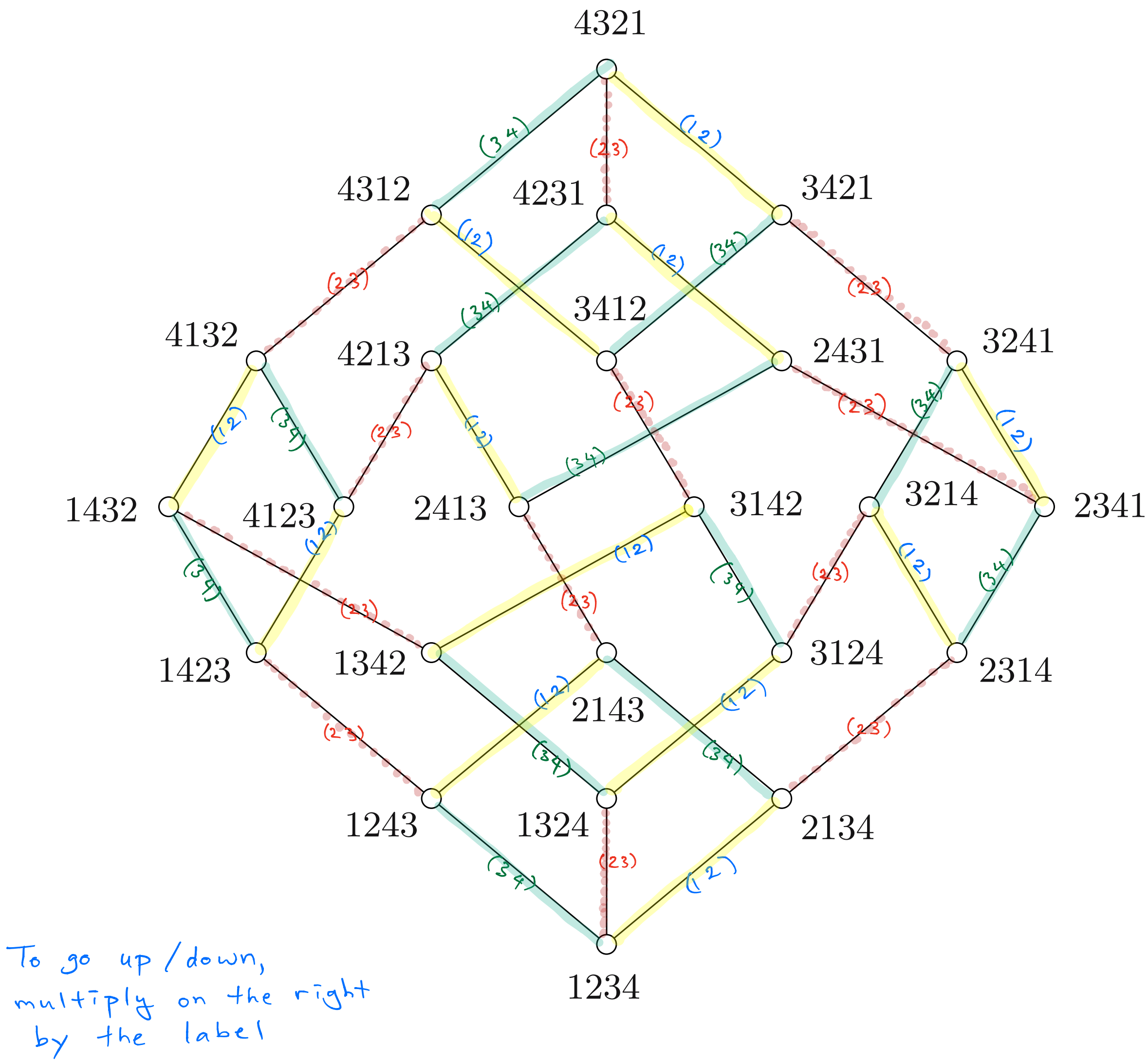

Figure 1: Weak order of S_4 from [BB05, Fig. 3.2, pg 67]

68 3. Weak order and reduced words

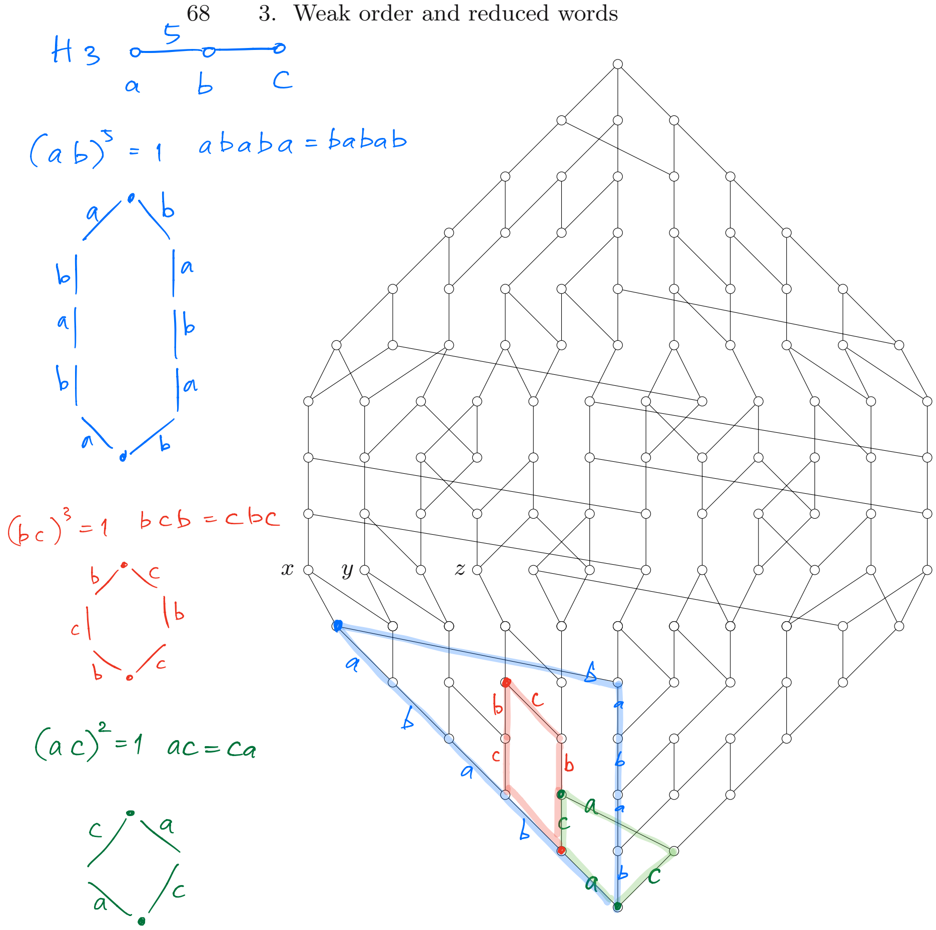

Figure 3.3. Weak order of H_3.

Figure 2: Weak order of H_3 from [BB05, Fig. 3.3, pg 68]