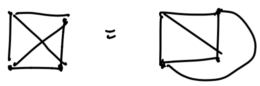
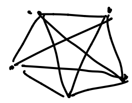
Plabic Graphs   PART II

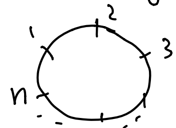
- Introduced in 2006 in a paper called "Total positivity, Grassmannians and networks" (by A. Postnikov), which has been cited 400+ times according to Google.

- Some applications outside of math (according to Wikipedia): quantum physics, computer vision (face and shape recognition), a data-visualization technique called grand tour.

Def A graph is <u>planar</u> if it can be drawn in the plane in such a way that the edges don't cross.

E.g. K_4  is planar, K_5  is non planar

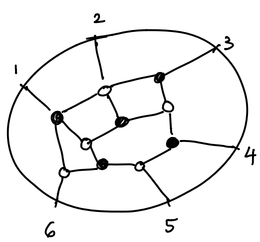
Def A <u>plabic</u> (planar bicolored) graph is a graph

- drawn inside a disk 
- has n boundary vertices on the boundary of the disk, labeled $1, 2, \ldots, n$ in clockwise order
- all internal vertices are colored using 2 colors (shaded / black and empty / white)

Assume simple graph (no multiple edges, no loops)

~~Assume no vertices of the same color are adjacent~~
Assume connected graph
Assume each boundary vertex i is adjacent to a single internal vertex.
Assume no leaf except for the boundary vertices.
degree 1 vertex

E.g.

D = 

Def "<u>Rules of the Road</u>"   Turn (maximally) right at black vertices ●
Turn (maximally) left at white vertices ○

Def Given a plabic graph, a <u>trip</u> T is a walk from a boundary vertex i which follows the "rules of the road" until it reaches a boundary vertex j. Refer to this trip as $T_{i \to j}$.

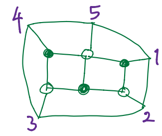
(New) **Def** A plabic graph is called <u>reduced</u> if every $T_{i\to j}$ is a path (as opposed to a closed path).

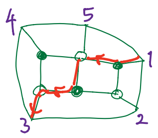
Def A permutation on $[n]=\{1,\dots,n\}$ is a bijection $[n]\to[n]$.

2-row notation $f=\begin{pmatrix}1&2&3&4&5\\3&4&5&1&2\end{pmatrix}$, $g=\begin{pmatrix}1&2&3&4&5\\1&4&3&2&5\end{pmatrix}$

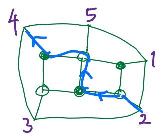
1-row notation $f = 3\ 4\ 5\ 1\ 2$, $g = 1\ 4\ 3\ 2\ 5$

Def Given a plabic graph G, define its <u>trip permutation</u>
$\pi_G = \pi(1)\cdots\pi(n)$ where $\pi(i)=j$ for each trip $T_{i\to j}$ of G

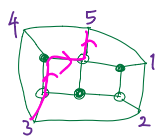
E.g. Let's compute the trip permutation π_{G_2} of $G_2=$ 



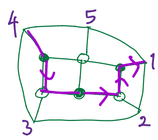
π_{G_2} sends 1 to 3



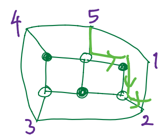
π_{G_2} sends 2 to 4



π_{G_2} sends 3 to 5



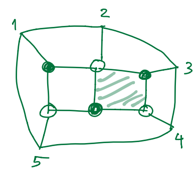
π_{G_2} sends 4 to 1



π_{G_2} sends 5 to 2

Hence $\pi_{G_2} = \begin{bmatrix}1&2&3&4&5\\3&4&5&1&2\end{bmatrix} = 34512$

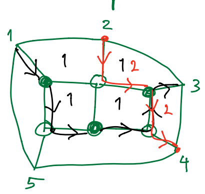
(in two line notation) (in one line notation)

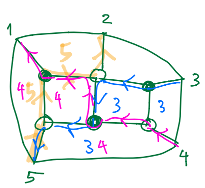
E.g. Last time, we computed the trip permutation of $G_1=$  to be also 34512.

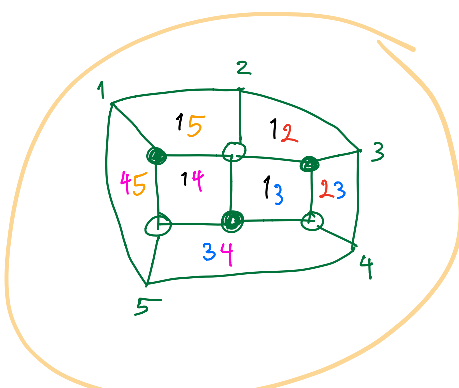
Def A (source) <u>face labeling</u> of G is the following map from the faces of G to the set of subsets of $[n]=\{1,2,\dots,n\}$.
For each trip $T_{i\to j}$, place the label i in every face which is to the left of $T_{i\to j}$.

E.x

G_1





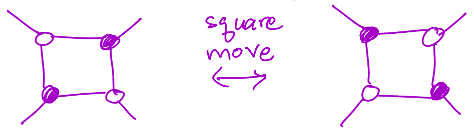


is the face labeling for G_1

LOCAL MOVES (M1'), (M2'), (M3)

(New) Def Local moves on plabic graphs

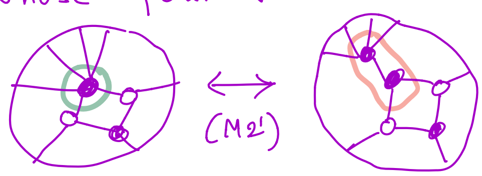
(M1') Square move:
If G has a square formed by four degree 3 vertices that are ALTERNATING IN COLORS, then we can switch the colors of these four vertices
~~(and add some degree 2 vertices to preserve the bipartiteness of the graph)~~



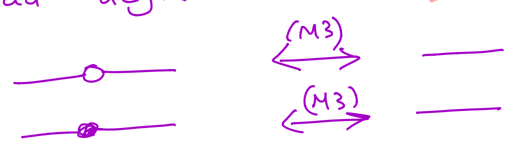
(M2') Edge contraction: Two adjacent vertices of the same color can be contracted into one vertex. This operation can be reversed.

Remark (M2') can be used to change any square face of G into a square face whose four vertices are degree 3 vertices

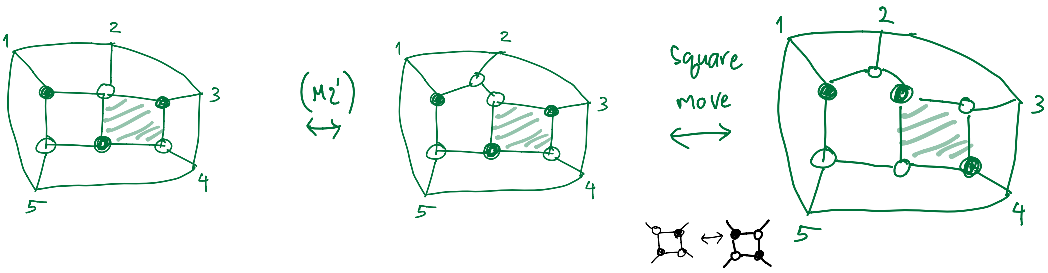
E.g.



(M3) Middle vertex insertion/removal.
We can remove or add degree 2 vertices, ~~as long as the graph remains bipartite~~.

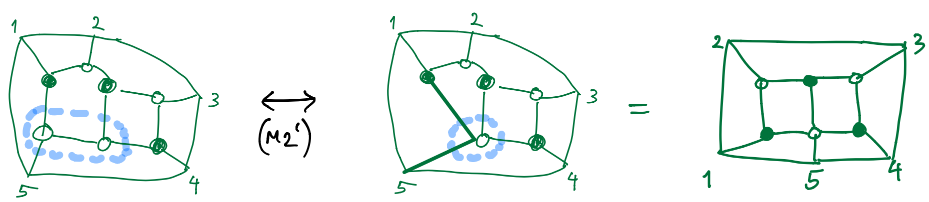


E.g. $G_1 :=$

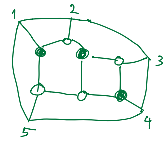


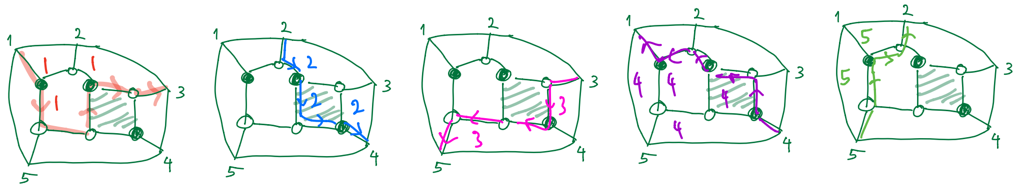
$=: G_2'$

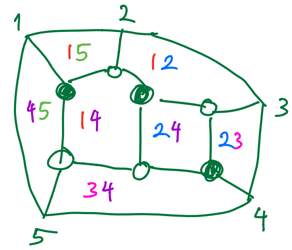
$G_2' =$

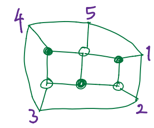


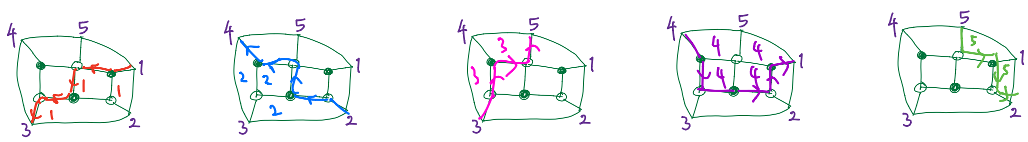
$=: G_2$

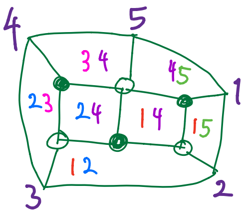
• Compute the (source) face labeling of $G_2' =$ 



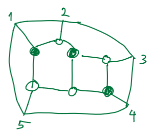


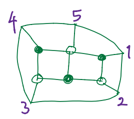
• Compare the face labeling of G_2' with the face labeling of $G_2 =$ 





Remark Move (M2') does not change the face labeling.

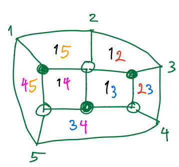
E.g. Face labeling of $G_2' =$  is the same as

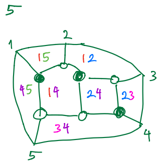
the face labeling of $G_2 =$ ,

and G_2 and G_2' differ by a move (M2')

Remark Move (M3) also does not change the face labeling.

Remark The square move (M1') DOES change the face labeling.

The face labeling  of G_1 is different from

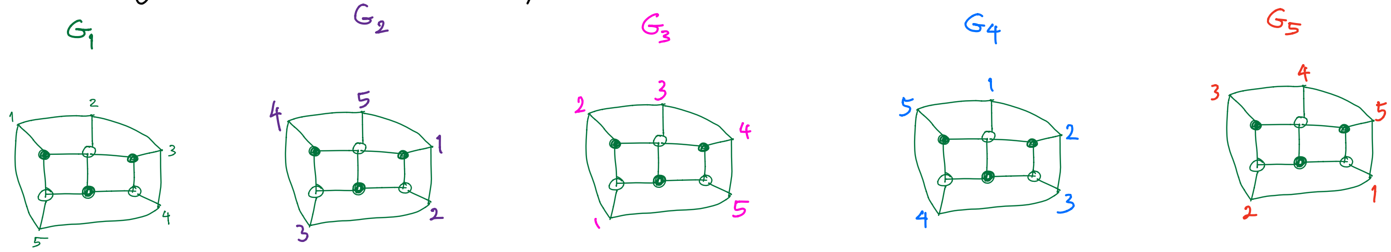
the face labeling  of G_2'.

Thm [Postnikov, Thm 13.4] If two plabic graphs G, H have the same trip permutation, then we can get from G to H by applying a sequence of the local moves (M1) – (M3).

Prop 2 The trip permutation is preserved by (M1'), (M2'), (M3)

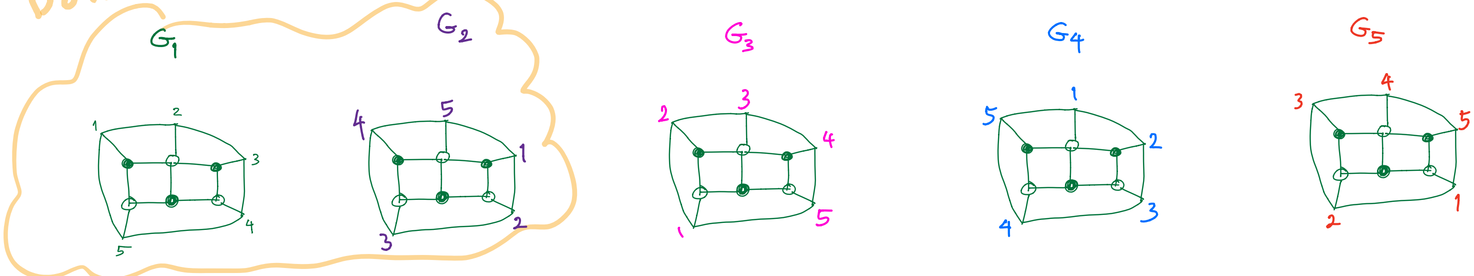
E.g. $\pi_{G_1} = \pi_{G_2'} = \pi_{G_2} = 34512$

Ex. These are the five plabic graphs with trip permutation 34512 (up to the local moves (M2') or (M3) which do not change the face labelings. For example, we say that G_2 and G_2' are equivalent because they have the same face labeling)



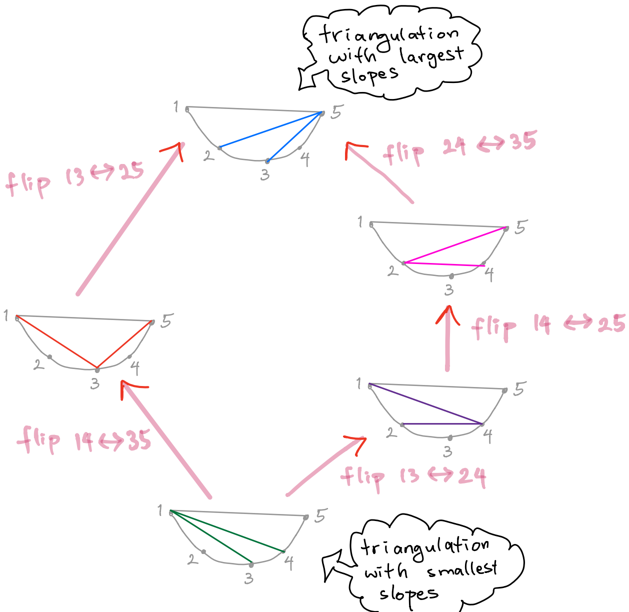
Example 3 Draw the face labeling of the following

Done above

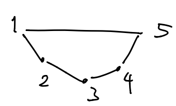


Thm (Scott) The plabic graphs with trip permutation $3456\ldots n12$ are Catalan objects.

HW 4' Below are the five triangulations of a 5-gon.



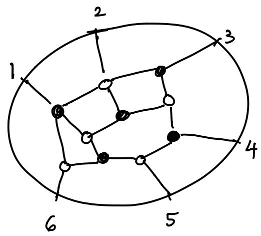
- A natural way to map the five plabic graphs (w/ trip permutation 34512) to these five triangulations is to map each face label ij to a line segment ij in the pentagon



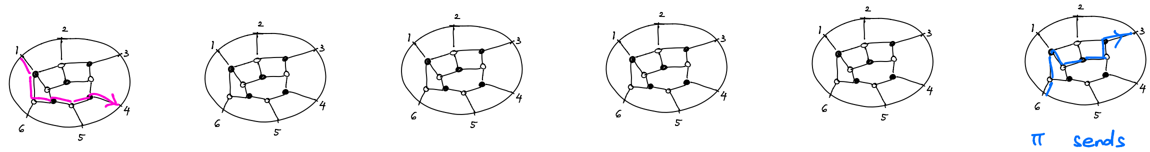
- The above rule for $T_1 \to T_2$ is that T_2 is the result of removing a diagonal of T_1 and replacing it with another diagonal of larger slope. This works for all n-gon.
- Can you think of a rule $G_1 \to G_2$ for the five plabic graphs in Example 5?

HW 5

Let D_1 =



Compute the trip permutation π of D_1. See p. 2 of this note.



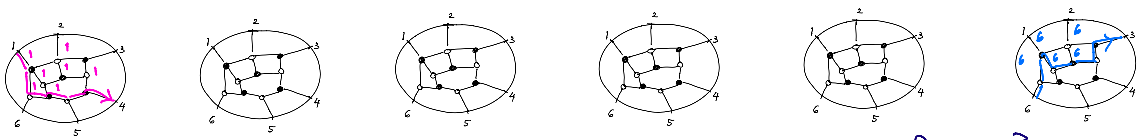
π sends 1 to 4

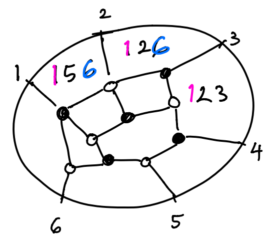
π sends 6 to 3

HW 6

- Compute the (source) face labeling of D_1. See p. 2 and 4 of this note.

Hint: Each face is labeled by three numbers.



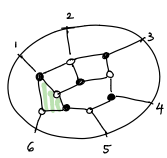


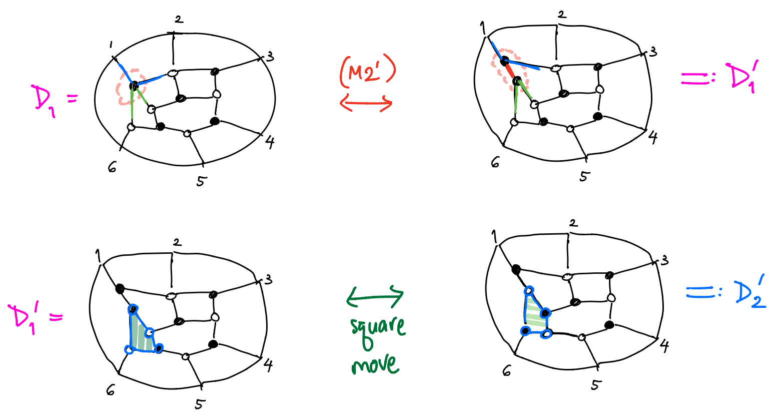
- What is special about the label of each external face?
- What is special about the label of each internal face?

HW 7

Let's apply (M2') so that we can apply the square move (M1') to D_1.

D_1 =





Compute the face labeling of D_2' and compare with the face labeling of D_1.

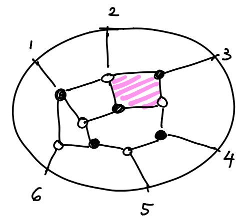
What changes and what stays the same?

HW 8

Follow the steps $D_1 \overset{(M2')}{\longleftrightarrow} D_1' \overset{\text{square move}}{\longleftrightarrow} D_2'$ given in HW 7 above,

but for the highlighted square

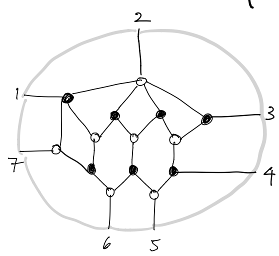
$D_1 =$

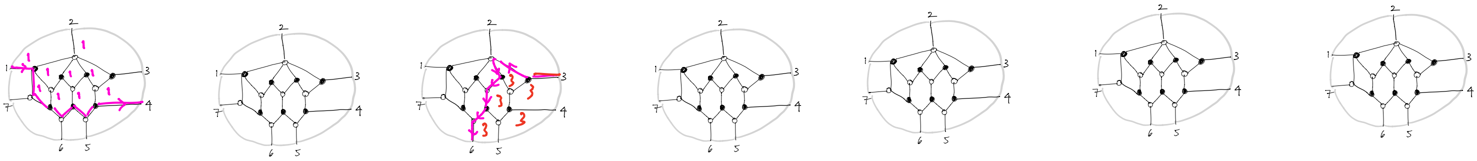


HW 9

Compute the trip permutation of E_1. See p. 2 of this note.

$E_1 =$





π sends 1 to 4

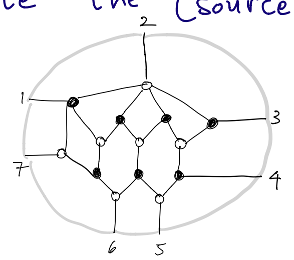
π sends 3 to 6

HW 10

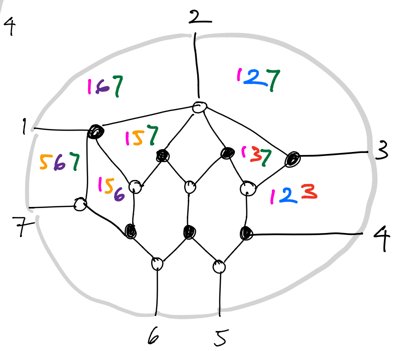
Compute the (source) face labeling of E_1. See p. 2 and 4 of this note.

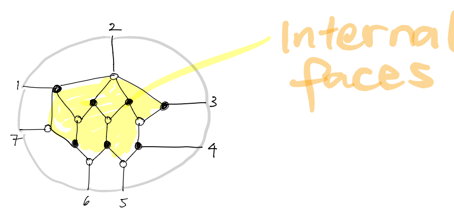
Hint: Each face is labeled by three numbers.

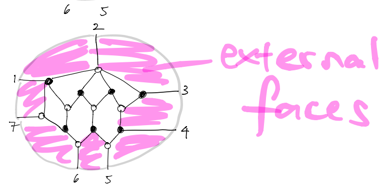
$E_1 =$



partially completed face labeling of E_1.







- What is special about the label of the external faces?
- What is special about the label of the internal faces?

the end

Ref J. Scott "Grassmannians and Cluster Algebras"
A. Postnikov "Total Positivity, Grassmannians, and Networks"