8.2.2 *Products of Exponential Generating Functions*

Combinatorics class
Bóna's textbook

Just as we have seen for ordinary generating functions, the product of two exponential generating functions has a very natural combinatorial meaning.

Lemma 8.20. *Let $\{a_i\}$ and $\{b_k\}$ be two sequences, and let $A(x) = \sum_{i\geq 0} a_i \frac{x^i}{i!}$ and $B(x) = \sum_{k\geq 0} b_k \frac{x^k}{k!}$ be their exponential generating functions. Define $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, and let $C(x)$ be the exponential generating function of the sequence $\{c_n\}$. Then*

$$A(x)B(x) = C(x).$$

In other words, the coefficient of $x^n/n!$ in $A(x)B(x)$ is $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$.

Lemma If $A(x) = \sum_{j=0}^{\infty} a_j \frac{x^j}{j!}$, $B(x) = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}$, $C(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \right) \frac{x^n}{n!}$

then $A(x)\,B(x) = C(x)$.

Proof. Just as in the proof of Lemma 8.4, multiplying $A(x)$ by $B(x)$ involves multiplying each term of $A(x)$ by each term of $B(x)$. A general term in this product is of the form

$$a_i \frac{x^i}{i!} \cdot b_j \frac{x^j}{j!} = a_i b_j \cdot \frac{x^{i+j}}{i!j!} \cdot \frac{(i+j)!}{(i+j)!} = a_i b_j \cdot \frac{x^{i+j}}{(i+j)!} \cdot \binom{i+j}{i}.$$

Such a product is of degree n if and only if $i + j = n$, and the statement follows. □

Proof of Lemma A general term of $A(x)B(x) = \left(\sum_{j=0}^{\infty} a_j \frac{x^j}{j!}\right)\left(\sum_{k=0}^{\infty} b_k \frac{x^k}{k!}\right)$

is of the form $\left(a_j \frac{x^j}{j!}\right)\left(b_k \frac{x^k}{k!}\right) = a_j b_k \frac{x^{j+k}}{j!\,k!}$

$$= a_j b_k \frac{x^{j+k}}{j!\,k!} \cdot \frac{(j+k)!}{(j+k)!}$$

$$= a_j b_k \frac{x^{j+k}}{(j+k)!} \frac{(j+k)!}{j!\,k!}$$

$$= a_j b_k \frac{x^{j+k}}{(j+k)!} \binom{j+k}{j}$$

Such product $\left(a_j \frac{x^j}{j!}\right)\left(b_k \frac{x^k}{k!}\right) = a_j b_k \frac{x^{j+k}}{(j+k)!} \binom{j+k}{j}$ is of degree n iff $j+k=n$.

So the coefficient of $\frac{x^n}{n!}$ in $A(x)B(x)$ is

$a_0 b_n \binom{n}{0} + a_1 b_{n-1} \binom{n}{1} + a_2 b_{n-2} \binom{n}{2} + \cdots + a_n b_0 \binom{n}{0} = \sum_{i=0}^{n} a_i b_{n-i} \binom{n}{i}$ [end of proof]

Theorem 8.21 (Product formula, exponential version). *Let a_n be the number of ways to build a certain structure on an n-element set, and let b_n be the number of way to build another structure on an n-element set. Let c_n be the number of ways to separate $[n]$ into the disjoint subsets S and T, $(S \cup T = [n])$, and then to build a structure of the first kind on S, and a structure of the second kind on T. Let $A(x)$, $B(x)$, and $C(x)$ be the respective exponential generating functions of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Then*

two subsets, not just two intervals

$$A(x)B(x) = C(x).$$

(OGF) (EGF)

Note that while Theorems 8.5 and 8.21 sound very similar, they apply in different circumstances. Theorem 8.5 applies when $[n]$ is split into two parts so that one part is $[i]$. That is, $[n]$ is split into *intervals*. Theorem 8.21 applies when $[n]$ is split into two parts with no restrictions. In other words, the first theorem applies when our objects are linearly ordered (like days in a calendar, or people in a line), and we cut that linear order somewhere to get two subsets. The second theorem applies when we are free to choose our two subsets, that is, they do not have to be consecutive objects in a previously ordered line.

Proof. (of Theorem 8.21) If S has i elements, then there are $\binom{n}{i}$ ways to choose the elements of S. Then there are a_i ways to build a structure of the first kind on S, and b_{n-i} ways to build a structure of the second kind on T, and this is true for all i, as long as $0 \leq i \leq n$. Therefore, $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, and our claim follows from Lemma 8.20. □

Example 8.22. A football coach has n players to work with at today's practice. First he splits them into two groups, and asks the members of each group to form a line. Then he asks each member of the first group to take on an orange shirt, or a white shirt, or a blue shirt. ~~Members of the other group keep their red shirt.~~ In how many different ways can all this happen?

Not relevant to the problem

Solution. Let us assume that the coach selects k people to form the first group. Let a_k be the number of ways these k people can take on an orange or white or blue shirt, and then form a line. Then $a_k = k!3^k$, so the exponential generating function of the sequence $\{a_k\}$ is

$$A(x) = \sum_{k\geq 0} k!3^k \frac{x^k}{k!} = \sum_{k\geq 0} \cancel{k!}3^k \frac{x^k}{\cancel{k!}} = \sum_{k=0}^{\infty} (3x)^k = \frac{1}{1-3x}.$$

Geometric series

Similarly, assume there are m people in the second group. Let b_m be the number of ways these m people can form a line. Then $b_m = m!$, and the exponential generating function of the sequence $\{b_m\}$ is

$$B(x) = \sum_{m\geq 0} m! \frac{x^m}{m!} = \frac{1}{1-x}.$$

Let c_n be the number of ways the players can follow the instructions of the coach, and let $C(x)$ be the exponential generating function of the sequence $\{c_n\}$. Then the Product formula implies

$$C(x) = A(x)B(x) = \boxed{\frac{1}{1-3x} \cdot \frac{1}{1-x}.}$$

The EGF for c_n →

$$\frac{1}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$$1 = A(1-x) + B(1-3x)$$

$x=1: \quad 1 = B(-2)$

$x=\frac{1}{3}: \quad 1 = A\,\frac{2}{3}$

$A = \frac{3}{2}$, $B = -\frac{1}{2}$

This leads to the partial fraction decomposition

$$C(x) = \frac{3}{2} \cdot \frac{1}{1-3x} - \frac{1}{2} \cdot \frac{1}{1-x}.$$

Therefore,

$$C(x) = \frac{3}{2} \sum_{n \geq 0} 3^n x^n - \frac{1}{2} \sum_{n \geq 0} x^n = \sum_{n \geq 0} \frac{1}{2} 3^{n+1} x^n - \sum_{n \geq 0} \frac{1}{2} x^n = \sum_{n \geq 0} \left(\frac{3^{n+1} - 1}{2} \right) x^n.$$

To find the closed formula for c_n, we do the following computation.

$$C(x) = \sum_{n \geq 0} \left(\frac{3^{n+1} - 1}{2} \right) x^n$$

$$= \sum_{n \geq 0} \boxed{\left(\frac{3^{n+1} - 1}{2} \right) n!} \frac{x^n}{n!}$$

So for all $n \geq 0$, the coefficient of $x^n/n!$ in $C(x)$ is $\boxed{c_n = n!(3^{n+1} - 1)/2.}$

[end of Ex 8.22]