$x/(1-x)$, and $B(x) = 1/(1-2x)$. So

$$G(x) = B(A(x)) = \frac{1}{1 - \frac{2x}{1-x}} = \frac{1-x}{1-3x} = \frac{1}{1-3x} - \frac{x}{1-3x},$$

$$G(x) = \sum_{n\geq 0} 3^n x^n - \sum_{n\geq 1} 3^{n-1} x^n = 1 + \sum_{n\geq 1} 2 \cdot 3^{n-1} x^n.$$

Consequently, if $n \geq 1$, the officer in charge has $2 \cdot 3^{n-1}$ options.

Quick Check

(1) Let $a_0 = 1$, and let $a_{n+1} = 3a_n - 1$ for $n \geq 0$. Find an explicit formula for a_n.

(2) Let b_n be the number of partitions of the integer n into even parts that are at most 6, and at most one odd part (of any size). Find an explicit formula for the ordinary generating function $B(x) = \sum_{n\geq 0} b_n x^n$.

(3) Let us revisit Example 8.16 with the additional restriction that the non-empty units that the officer forms cannot consist of more than three people each. Let g_n be the total number of the ways in which the officer can proceed. Find an explicit formula for the ordinary generating function $G_n(x) = \sum_{n\geq 0} g_n x^n$.

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8.2 Exponential Generating Functions

8.2.1 *Recurrence Relations and Exponential Generating Functions*

Not all recurrence relations can be turned into a closed formula by using an ordinary generating function. Sometimes, a closed formula may not exist. Some other times, it could be that we have to use a different kind of generating function.

Example 8.17. Let $a_0 = 1$, and let $a_{n+1} = (n+1)(a_n - n + 1)$, if $n \geq 0$. Find a closed formula for a_n.

If we try to solve this recurrence relation by ordinary generating functions, we run into trouble. The reason for this is that this sequence grows too fast, and its ordinary generating function will therefore not have a closed form. Let us instead make the following definition.

Definition 8.18. Let $\{f_n\}_{n\geq 0}$ be a sequence of real numbers. Then the formal power series $\boxed{F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}}$ is called the *exponential generating function* of the sequence $\{f_n\}_{n\geq 0}$.

The word "exponential" is due to the fact that the exponential generating function of the constant sequence $f_n = 1$ is e^x. Let us use this new kind of generating function to solve the example at hand.

Solution. (of Example 8.17.) Let $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ be the exponential generating function of the sequence $\{a_n\}_{n\geq 0}$. From this point on, we proceed in a way that is very similar to the method of the previous section. Let us multiply both sides of our recursive formula by $x^{n+1}/(n+1)!$, and sum over all $n \geq 0$ to get

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} (n-1) \frac{x^{n+1}}{n!}. \tag{8.15}$$

Note that the left-hand side is $A(x)-1$, while the first term of the right-hand side is $xA(x)$. This leads to

$$A(x) - 1 = xA(x) - x^2e^x + xe^x,$$

$$\boxed{A(x) = \frac{1}{1-x} + xe^x} = \sum_{n\geq 0} x^n + \sum_{n\geq 0} \frac{x^{n+1}}{n!}.$$

The coefficient of $x^n/n!$ in $\sum_{n\geq 0} x^n$ is $n!$, while the coefficient of $x^n/n!$ in $\sum_{n\geq 0} \frac{x^{n+1}}{n!}$ is n. Indeed, this second term has summand $x^n/(n-1)!$. Therefore, the coefficient of $x^n/n!$ in $A(x)$ is $\boxed{a_n = n! + n}$.

Ex 8.17.

$a_0 = 1$

$a_{n+1} = (n+1)\, a_n - (n+1)(n-1)$ for $n \geq 0$

(i) Find the exponential generating function (EGF) for a_n

(ii) Find a closed formula for a_n

Solution

(i) Let $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \ldots$

$$a_{n+1} = (n+1)\, a_n - (n+1)(n-1) \quad \text{(Recursive formula)}$$

$$a_{n+1} \frac{x^{n+1}}{(n+1)!} = (n+1) a_n \frac{x^{n+1}}{(n+1)!} - (n+1)(n-1) \frac{x^{n+1}}{(n+1)!} \quad \left(\text{Multiply by } \frac{x^{n+1}}{(n+1)!}\right)$$

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} (n+1) a_n \frac{x^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} (n+1)(n-1) \frac{x^{n+1}}{(n+1)!} \quad \left(\text{Sum over all } n \geq 0\right)$$

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} \cancel{(n+1)} a_n \frac{x^{n+1}}{\underset{n!}{\cancel{(n+1)!}}} - \sum_{n=0}^{\infty} \cancel{(n+1)}(n-1) \frac{x^{n+1}}{\underset{n!}{\cancel{(n+1)!}}} \quad (\text{Simplify})$$

$$\sum_{n=0}^{\infty} a_{n+1} \frac{x^{n+1}}{(n+1)!} \overset{(*)}{=} \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} (n-1) \frac{x^{n+1}}{n!} \quad \left(\text{After you simplify}\right)$$

- Note the LHS is $a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + \ldots$ So the LHS is $A(x) - a_0 = \boxed{A(x) - 1}$
- Note the first term of the RHS is $\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} = x \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \boxed{x\, A(x)}$
- Note the second term of the RHS is

$$-\sum_{n=0}^{\infty} (n-1) \frac{x^{n+1}}{n!} = -\sum_{n=0}^{\infty} n \frac{x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

because $0 \cdot \frac{x^{0+1}}{0!} = 0$

$$= -\sum_{n=1}^{\infty} n \frac{x^{n+1}}{n!} + x \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= -\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n-1)!} + x \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} + x \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= -x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} + x \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \boxed{-x^2 e^x + x e^x} \qquad \left(\text{since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)$$

So eq. (*) is equivalent to $A(x) - 1 = x A(x) - x^2 e^x + x e^x$

$$A(x) - x A(x) = 1 - x^2 e^x + x e^x$$

$$A(x)(1 - x) = 1 - x^2 e^x + x e^x$$

$$A(x) = \frac{1 - x^2 e^x + x e^x}{1 - x}$$

$$= \frac{1 + x e^x (-x + 1)}{1 - x}$$

This is the exponential generating function (EGF) for a_n, the answer to part (i).

$$= \boxed{\frac{1}{1-x} + x e^x}$$

(ii) To find a closed formula for a_n, use part (i)

$$A(x) = \frac{1}{1-x} + x e^x$$

$$= \sum_{n=0}^{\infty} x^n + x \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \left(\text{since } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ and } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} \qquad \left(\text{Change the index so that we use } x^n \text{ instead of } x^{n+1} \right)$$

$$= \sum_{n=0}^{\infty} n! \frac{x^n}{n!} + \sum_{n=1}^{\infty} n \frac{x^n}{n!} \qquad \text{Rewrite in such a way that we see the } \frac{x^n}{n!} \text{ factor}$$

$$= \sum_{n=0}^{\infty} n! \frac{x^n}{n!} + \sum_{n=0}^{\infty} n \frac{x^n}{n!} \qquad \left(\text{the equality holds because } 0 \cdot \frac{x^0}{0!} = 0 \right)$$

$$= \sum_{n=0}^{\infty} (n! + n) \frac{x^n}{n!}$$

Since the coefficient of $\frac{x^n}{n!}$ in $A(x)$ is $n! + n$,

we have $a_n = \boxed{n! + n}$

[end of Ex. 8.17]

Example 8.19. Let $f_0 = 0$, and let $f_{n+1} = 2(n+1)f_n + (n+1)!$ if $n \geq 0$. Find an explicit formula for f_n.

Solution. Let $F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ be the exponential generating function of the sequence f_n. Let us multiply both sides of our recursive formula by $x^{n+1}/(n+1)!$, then sum over all $n \geq 0$. We get

$$\sum_{n\geq 0} f_{n+1}\frac{x^{n+1}}{(n+1)!} = 2x\sum_{n\geq 0} f_n \frac{x^n}{n!} + \sum_{n\geq 0} x^{n+1}. \tag{8.16}$$

As $f_0 = 0$, the left-hand side of (8.16) is equal to $F(x)$, while the first term of the right-hand side is $2xF(x)$, and the second term of the right-hand side is $x/(1-x)$. Therefore, we get

$$F(x) = 2xF(x) + \frac{x}{1-x},$$

Explain the computation in details as I have done for Ex 8.17

$$F(x) = \frac{x}{(1-x)(1-2x)}.$$

Therefore,

$$F(x) = \sum_{n\geq 0}(2^n - 1)x^n,$$

and so the coefficient of $x_n/n!$ in $F(x)$ is $f_n = (2^n - 1)n!$.

Note: Use Partial Fraction Decomposition to get $\frac{x}{(1-x)(1-2x)} = \frac{1}{1-2x} - \frac{1}{1-x}$