Sec 8.1.3 Compositions of Generating Functions

Let $F(x) := \frac{1}{1-x} = 1 + x + x^2 + \dots$, $G(x) = g_0 + g_1 x + g_2 x^2 + \dots$

Want to compose F with G

$$F(G(x)) \overset{?}{=} \frac{1}{1-G(x)} = 1 + G(x) + G(x)^2 + G(x)^3 + \dots$$

can we ?

$$= 1 + (g_0 + g_1 x + g_2 x^2 + \dots) + (g_0 + g_1 x + g_2 x^2 + \dots)^2 + (g_0 + g_1 x + g_2 x^2 + \dots)^3 + \dots$$

infinite sums

$$= (1 + g_0 + g_0^2 + g_0^3 + \dots) + (g_1 + \underline{g_0 g_1 + g_1 g_0} + \underline{g_0 g_0 g_1 + g_0 g_1 g_0 + g_1 g_0 g_0} + \dots) x^1 + \dots$$

$$+ (g_2 + \underline{g_0 g_2 + g_2 g_0 + g_1 g_1} + \underline{3 g_0 g_0 g_2 + 3 g_0 g_1 g_1} + \dots) x^2 + \dots$$

doesn't make sense if $g_0 \neq 0$. "unless those infinite sums converge... But we don't think about convergence"

If $g_0 = 0$, $F(G(x)) \overset{\text{def } 8.12}{=} 1 + G(x) + (G(x))^2 + (G(x))^3 + \dots + (G(x))^k + \dots$

Claim:

We will be able to compute the coefficient of, say, $k=5$, from just the terms $G(x) + (G(x))^2 + \dots + (G(x))^5$

Why? Every single term $(G(x))^k = (g_1 x + g_2 x^2 + g_3 x^3 + \dots)^k$

$$= [x (g_1 + g_2 x + g_3 x^2 + \dots)]^k$$

$$= x^k (g_1 + g_2 x + g_3 x^2 + \dots)^k \quad \text{with } k > 5$$

will contain only powers of x higher than the 5th, & so we won't need to look at those terms to find the coefficient of x^5

$$\text{If } g_0 = 0, \quad F(G(x)) = 1 + G(x) + (G(x))^2 + (G(x))^3 + \dots + (G(x))^k + \dots$$

$$= 1 + g_1 x + (g_2 + g_1 g_1) x^2 + (g_1 g_1 g_1 + g_1 g_2 + g_2 g_1 + g_3) x^3 + \dots$$

$$+ \left(\sum_{g_{i_1} + \dots + g_{i_k} = k} g_{i_1} g_{i_2} \cdots g_{i_k} \right) x^k + \dots$$

The computation of the coefficient of each x^k for $F(G(x))$ is a finite process, so the coefficient of each x^k is well-defined, so the series $F(G(x))$ is well defined.

E.g. $e^{[e^x-1]}$ is a well defined formal series because

$F(x) := e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$G(x) := e^x - 1 = 0 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ here $g_0 = 0$

but

$e^{[e^x]}$ is not defined (from our definition of composition of power series)

because if $G(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ then $g_0 = 1 \neq 0$.

Thm 8.13 Let $a_0 := 0$, &

let a_n be the # of ways to build a structure on an n-elt set

Let $h_0 = 1$, &

let h_n be the # of ways to split $[n]$ into an unspecified number of disjoint, non-empty intervals,

then build that structure on each of these intervals.

Then $H(x) = \frac{1}{1 - A(x)} \overset{\text{def } 8.12}{=} 1 + A(x) + A(x)^2 + \cdots$

where $A(x) = \sum_{n \geq 0} a_n x^n$, $H(x) = \sum_{n \geq 0} h_n x^n$

Pf Thm 8.5 (Product Formula) says that $A(x)^k = \overbrace{A(x) \cdot A(x) \cdot \cdots \cdot A(x)}^{k \text{ times}}$ is the generating function for the # of ways to split $[n]$ into k intervals, then build the same structure on each interval.

So the number of ways to split $[n]$ into an unspecified number of intervals, then build the same structure on each interval is $A(x) + A(x)^2 + A(x)^3 + \cdots + A(x)^k + \cdots = \sum_{k \geq 1} A(x)^k$.

Since $a_0 = 0$, none of $A(x)^k$ has a nonzero constant term.

But $h_0 = 1$ by def, so $H(x) = 1 + \sum_{k \geq 1} A(x)^k = \frac{1}{1 - A(x)}$. $\boxed{8.13}$

See the problem in the book.

Ex 8.14 a_n is the # of ways to choose a leader from an n-elt set, so $a_n = n$.

Then $A(x) = \sum_{n=1}^{\infty} n x^n$

$H(x) = \sum_{n\geq 0} h_n x^n \overset{\text{thm}}{=} 1 + (1x + 2x^2 + 3x^3 + \dots) + (2+1)x^2 + (1+2+2+3)x^3 + \dots$

Check: For $n=1$, split into one interval

For $n=2$, split into one interval of size 2 (2 choices)

two intervals of size 1 each (1 choice)

For $n=3$, can split

1, 1, 1 or 1, 2 or 2, 1 or 3 total: 8

1 choice, 2 choices, 2 choices, 3 choices

Then
$$A(x) = \sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} = x\left(\sum_{n=0}^{\infty} x^n\right)' = x\left(\frac{1}{1-x}\right)' = x\frac{1}{(1-x)^2}$$

$$H(x) = \frac{1}{1-A(x)} = \frac{1}{1-\left[\frac{x}{(1-x)^2}\right]} = \frac{1}{\left[\frac{(1-x)^2 - x}{(1-x)^2}\right]} = \frac{(1-x)^2}{1^2 - 2x + x^2 - x} = \frac{(1-x)^2}{1-3x+x^2}$$

$$= \frac{1-2x+x^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2}$$

$$\frac{1}{1-3x+x^2} \overset{\text{quadratic formula}}{=} \frac{1}{(x-\alpha)(x-\beta)} \quad \text{where } \alpha = \frac{3+\sqrt{5}}{2}, \ \beta = \frac{3-\sqrt{5}}{2}$$

$$= \frac{A}{x-\alpha} + \frac{B}{x-\beta} \quad \text{Do Partial Fraction}$$

$$H(x) = 1 + \frac{1}{\sqrt{5}}\left(\sum_{k=1}^{\infty} \alpha^n x^n - \sum_{k=1}^{\infty} \beta^n x^n\right)$$

$$\therefore \ h_n = \begin{cases} 1 & \text{if } n=0 \\ \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \end{cases}$$

$h_n = \{1, 1, 3, 8, 21, 55, \dots\}$ We get every other Fibonacci #s

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Thm 8.15 (Compositional Formula)

Let $a_0 = 0$, & ← if $a_0 \neq 0$ we cannot define $B(A(x))$

a_n be the # of ways to build a structure on an n-elt set.

Let $b_0 = 1$, &

b_n be the # of ways to build a second structure on an n-elt set.

let $g_0 = 1$, &

g_n be the # of ways to split the set $[n]$ into an unspecified number of non-empty intervals,
build a structure of the 1st kind on each of these intervals,
& then build a structure of the second kind on the set of the intervals.

Denote by $A(x)$, $B(x)$, $G(x)$ the gen functions of the sequences $\{a_n\}$, $\{b_n\}$, $\{g_n\}$.

Then $G(x) = B(A(x))$.

Pf

For a given k, if we split $[n]$ into k intervals,

there are b_k ways to build a structure of the 2nd kind

on the k-elt set (set of these k intervals).

The product formula says the OGF for the # of ways

to build a structure of the 1st kind on each interval

is $\underbrace{A(x)\,A(x)\,\cdots\,A(x)}_{k} = A(x)^k$.

The contribution of this case (when $[n]$ is split into k intervals)

is $b_k\,(A(x))^k$. Summing over all k, we get

$$G(x) = \sum_{k=0}^{\infty} b_k (A(x))^k \stackrel{\text{def}}{=} B(A(x)).$$