The famous Catalan #s (Ref: 8.1.2.1 or google "Catalan #s")

Object 1: Grouping with n parentheses

n = 1 ( )

n = 2 (( )) , ( )( )

n = 3 ((( ))), ( )(( )), (( ))( ), (( )( )), ( )( )( )

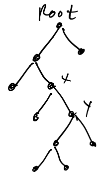
Object 2: Full binary tree with n parent vertices

Def A full binary tree is a tree with a distinguished vertex called the root s.t every parent vertex has exactly two children.

E.g.

Notation: x is the parent of y, and y is a child of x

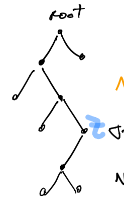
Root



x y

yes

Root



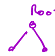
NO

z

No

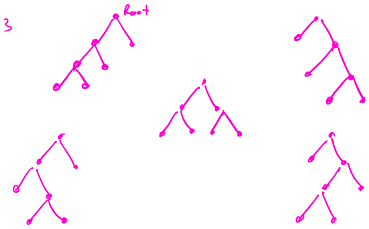
because z is a parent vertex but z has only one child.

n = 1 Root



n = 2 Root

n = 3 Root



See also: "binary trees with n vertices" from Reading HW 9

Object 3: Triangulation of an (n+2)-gon

n = 1

n = 2

1 2 3 4

n = 3

n = 4

...

object 4

$a_0 = 1$
a_1
a_2 ..
$a_n = 1$ s.t $a_k \mid a_{k-1} + a_{k+1}$ $\forall\, k = 1, \dots, n-1$.

n=2: 1, 2

n=3: $\begin{smallmatrix}1\\1\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\3\end{smallmatrix}$, $\begin{smallmatrix}3\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\1\end{smallmatrix}$

n=4: 1 1 1, 1 1 2,

object 4 : For $n \geqslant 2$, a tuple $(a_0 = 1, a_1, a_2, \dots, a_{n-1}, a_n = 1)$ of positive integers is called admissible if

$$a_k \text{ divides } (a_{k-1} + a_{k+1}) \text{ for all } k = 1, 2, \dots, n-1.$$

$n = 2$ $\quad (a_0 = 1,\ a_1 = 1,\ a_2 = 1)$ and $(a_0 = 1,\ a_1 = 2,\ a_2 = 1)$

$n = 3$ $\quad (a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_2 = 1)$

$(1,\ 1,\ 2,\ 1)$

$(1,\ 2,\ 1,\ 1)$

$(1,\ 2,\ 3,\ 1)$

$(1,\ 3,\ 2,\ 1)$

$n = 4$ $\quad (1,\ 3,\ 5,\ 2,\ 1)$, etc $\quad$ total is 14

On board in groups

Pick your favorite objects & draw the objects for $n=4$

Find bijections between the objects

Problem Let $C_0 = 1$ and

let h_n denote the # of triangulations of an $(n+2)$-gon for $n \geq 1$.

① Prove that "the rec. rel $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \dots + C_{n-1} C_0$ for $n \geq 1$ holds"

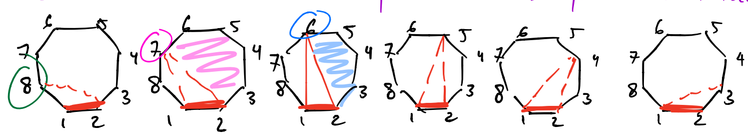
Answer

Compute $C_1 = 1$, △ and $C_1 = C_0 C_0$ is satisfied.

Let $n \geq 2$. Distinguish one side of the $(n+2)$-gon P and call it the base.

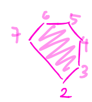
In any triangulation of P, the base forms one side of a triangle.

E.g. if $n = 6$ there are 6 possibilities for the triangle which the base is a side of.

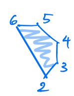


If the third corner is labeled 8, then triangulate the remaining 7-gon in C_5 ways.
$5+2$

If the third corner is labeled 7, then triangulate  in C_1 way

and triangulate the 6-gon  in C_4 ways, for a total of $C_1 C_4$ triangulations.

If the third corner is labeled 6, then triangulate the 4-gon  in C_2 ways and

triangulate the 5-gon  in C_3 ways, for a total of $C_2 C_3$ triangulations.

Continuing this produces $C_6 = C_5 + C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 + C_5$.

In general, the same idea gives

$$C_n = C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \dots + C_{n-2} C_1 + C_{n-1} \quad \text{for } n \geq 1.$$

Since $C_0 = 1$, we can write $C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$ □

② Use this recurrence relation to compute the generating function $F(x) = \sum_{n=0}^{\infty} C_n x^n$

Ans Multiply the recurrence relation by x^n & sum over all $n \geq 1$

$$\sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} C_k C_{n-1-k} \right) x^n \qquad \text{LHS is } F(x) - C_0 = F(x) - 1.$$

RHS is $\underbrace{C_0 C_0 x^1}_{\text{for } n=1} + \underbrace{(C_0 C_1 + C_1 C_0) x^2}_{\text{for } n=2} + \underbrace{(C_0 C_2 + C_1 C_1 + C_2 C_0) x^3}_{\text{for } n=3} + \dots$

$$= x \left[C_0 C_0 + (C_0 C_1 + C_1 C_0) x + (C_0 C_2 + C_1 C_1 + C_2 C_0) x^2 + \dots \right]$$

$$= x \left[F(x) \right]^2$$

since $\boxed{\left(\sum_{n=0} C_n x^n \right) \left(\sum_{n=0} C_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n \text{ by Sec 8.1.2 Lemma} \quad = C_0 C_0 + (C_0 C_1 + C_1 C_0) x + (C_0 C_2 + C_1 C_1 + C_2 C_0) x^2 + \dots}$

$$\therefore \quad F(x) - 1 = x \left[F(x) \right]^2$$

$$0 = x \left[F(x) \right]^2 - F(x) + 1$$

+ or − ?

So $F(x) = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{1 \pm \sqrt{1 - 4x}}{2x}$

By def, $\sum_{n=0}^{\infty} c_n x^n \Big|_{x=0} = c_0 = 1$ "so we need $F(x)$ to have constant term 1".

$$\lim_{x\to 0^+} \frac{1+\sqrt{1-4x}}{2x} = +\infty$$

$1+\sqrt{1-4x} \to 2$ as $x \to 0$

$2x \to 0$ as $x \to 0^+$

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x}}{2x} = \lim_{x\to 0} \frac{-\frac{1}{2}(1-4x)^{-\frac{1}{2}}(-4)}{2}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{1-4x}} = 1$$

$\therefore \boxed{F(x) = \frac{1-\sqrt{1-4x}}{2x}}$ $\square$

(3) Find an explicit formula for C_n.

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4)^n x^n \quad \text{by Binomial Thm}$$

Compute $\binom{\frac{1}{2}}{n}$:

$\binom{\frac{1}{2}}{0} = 1$, $\binom{\frac{1}{2}}{1} = \frac{1}{2}$

$\frac{1}{2} - \frac{2n}{2} + \frac{2}{2} = \frac{-2n+3}{2}$

· If $n \geq 2$, then
$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}$$
$$= \frac{(-1)\cdot(-3)\cdot(-5)\cdots(-2n+3)}{2^n\, n!}$$
$$= (-1)^{n-1}\frac{(1)(3)(5)\cdots(2n-3)}{2^n\, n!}$$
$$= (-1)^{n-1}\frac{(2n-3)!!}{2^n\, n!}$$

Note: Semifactorial, $k!!$ is the product of all odd integers from 1 to $2n-3$.

$$\therefore \sqrt{1-4x} = 1 - 2x + \sum_{n\geq 2} (-1)^{n-1}\frac{(2n-3)!!}{2^n\, n!}(-4x)^n$$

$$= 1 - 2x - \sum_{n\geq 2} \frac{(2n-3)!!\, 2^n}{n!} x^n$$ because $(-1)^{n-1}(-1)^n = (-1)^{2n-1} = -1$ and $\frac{4^n}{2^n} = 2^n$

Rem
$$\frac{2^n(2n-3)!!}{n!}\cdot\frac{(n-1)!}{(n-1)!} = \frac{2}{n}\cdot\frac{(2n-3)!!\, 2^{n-1}(n-1)!}{(n-1)!\,(n-1)!}$$
$$= \frac{2}{n}\cdot\frac{(2n-2)!}{(n-1)!\,(n-1)!}$$
$$= \frac{2}{n}\binom{2n-2}{n-1}$$

because $\underbrace{2\cdot 2\cdot 2\cdots 2}_{n-1}\cdot(n-1)! = 2(n-1)\cdot 2(n-2)\cdots(2)$

$$\therefore \sqrt{1-4x} = 1 - 2x - 2\sum_{n\geq 2} \frac{1}{n}\binom{2n-2}{n-1} x^n.$$

$$F(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{\cancel{1}}{2x} - \frac{1}{2x}\left(\cancel{1} - 2x - 2\sum_{n=2}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n\right)$$
$$= 1 + \frac{1}{x}\sum_{n=2}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n$$
$$= 1 + \sum_{n=2}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^{n-1}$$
$$= 1 + \sum_{n=1}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$
$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$

$$\therefore \boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

$n = 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$
$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$