6. Prove that every closed odd walk in a graph contains an odd cycle.

7. Draw a connected graph having at most 10 vertices that has at least one cycle of each length from 5 through 9, but has no cycles of any other length.

8. Let P_1 and P_2 be two paths of maximum length in a connected graph G. Prove that P_1 and P_2 have a common vertex.

9. Let G be a graph of order n that is not connected. What is the maximum size of G?

10. Let G be a graph of order n and size strictly less than $n - 1$. Prove that G is not connected.

11. Prove that an edge e is a bridge of G if and only if e lies on no cycle of G.

12. Prove or disprove each of the following statements.

    (a) If G has no bridges, then G has exactly one cycle.

    (b) If G has no cut vertices, then G has no bridges.

    (c) If G has no bridges, then G has no cut vertices.

13. Prove or disprove: If every vertex of a connected graph G lies on at least one cycle, then G is 2-connected.

14. Prove that every 2-connected graph contains at least one cycle.

15. Prove that for every graph G,

    (a) $\kappa(G) \le \delta(G)$;

    (b) if $\delta(G) \ge n - 2$, then $\kappa(G) = \delta(G)$.

"Textbook: Combinatorics and Graph Theory" by H, H, M

16. Let G be a graph of order n.

    (a) If $\delta(G) \ge \frac{n-1}{2}$, then prove that G is connected.

    (b) If $\delta(G) \ge \frac{n-2}{2}$, then show that G need not be connected.

Sec 1.1.3 Special Types of Graphs

until we meet again . . .

— from *An Irish Blessing*

In this section we describe several types of graphs. We will run into many of them later in the chapter.

1. Complete Graphs

   We introduced complete graphs in the previous section. A complete graph of order n is denoted by K_n, and there are several examples in Figure 1.11.

edges is $\binom{n}{2}$

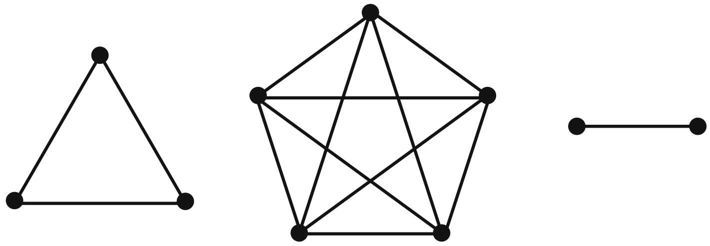


FIGURE 1.11. Examples of complete graphs.

2. Empty Graphs

   The *empty graph* on n vertices, denoted by E_n, is the graph of order n where E is the empty set (Figure 1.12).

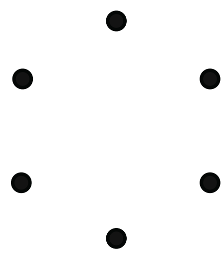


FIGURE 1.12. An empty graph.

3. Complements

   Given a graph G, the *complement* of G, denoted by $\overline{G}$, is the graph whose vertex set is the same as that of G, and whose edge set consists of all the edges that are *not* present in G (Figure 1.13).

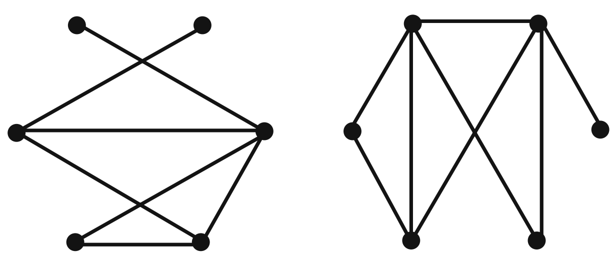


FIGURE 1.13. A graph and its complement.

4. Regular Graphs

   A graph G is *regular* if every vertex has the same degree. G is said to be *regular of degree* r (or r-*regular*) if $\deg(v) = r$ for all vertices v in G. Complete graphs of order n are regular of degree $n-1$, and empty graphs are regular of degree 0. Two further examples are shown in Figure 1.14.

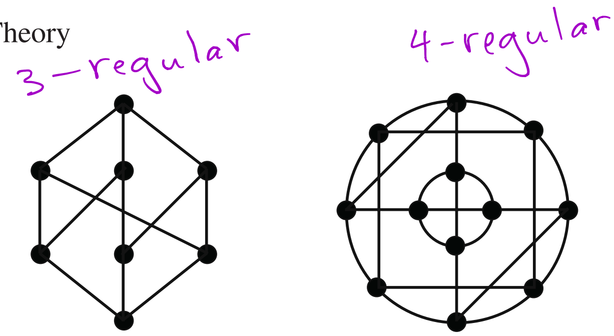


FIGURE 1.14. Examples of regular graphs.

5. Cycles

   The graph C_n is simply a cycle on n vertices (Figure 1.15).

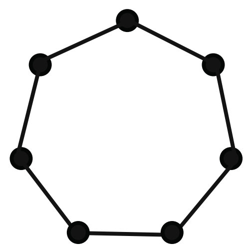


FIGURE 1.15. The graph C_7.

6. Paths

   The graph P_n is simply a path on n vertices (Figure 1.16).

FIGURE 1.16. The graph P_6.

7. Subgraphs

   A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we write $H \subseteq G$, and we say that G contains H. In a graph where the vertices and edges are unlabeled, we say that $H \subseteq G$ if the vertices *could* be labeled in such a way that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In Figure 1.17, H_1 and H_2 are both subgraphs of G, but H_3 is not.

8. Induced Subgraphs

   Given a graph G and a subset S of the vertex set, the *subgraph of G induced by* S, denoted $\langle S \rangle$, is the subgraph with vertex set S and with edge set $\{uv \mid u, v \in S \text{ and } uv \in E(G)\}$. So, $\langle S \rangle$ contains all vertices of S and all edges of G whose end vertices are *both* in S. A graph and two of its induced subgraphs are shown in Figure 1.18.

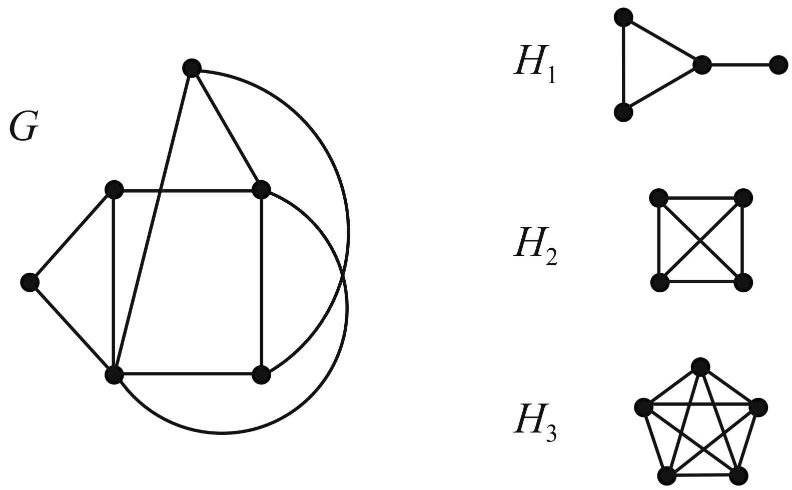


FIGURE 1.17. H_1 and H_2 are subgraphs of G.

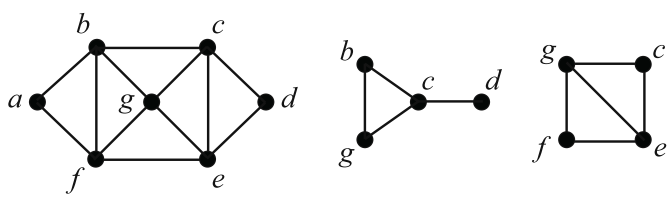


FIGURE 1.18. A graph and two of its induced subgraphs.

9. Bipartite Graphs

A graph G is *bipartite* if its vertex set can be partitioned into two sets X and Y in such a way that every edge of G has one end vertex in X and the other in Y. In this case, X and Y are called the *partite sets*. The first two graphs in Figure 1.19 are bipartite. Since it is not possible to partition the vertices of the third graph into two such sets, the third graph is not bipartite.

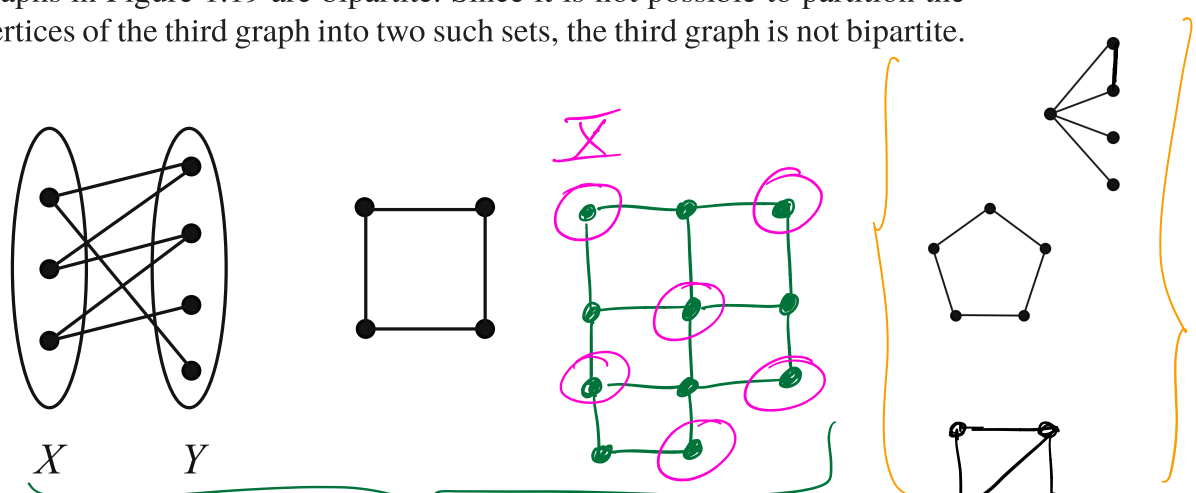


FIGURE 1.19. Bipartite graphs

Non-bipartite graphs

A bipartite graph with partite sets X and Y is called a *complete bipartite graph* if its edge set is of the form $E = \{xy \mid x \in X, y \in Y\}$ (that is, if every possible connection of a vertex of X with a vertex of Y is present in the graph). Such a graph is denoted by $K_{|X|,|Y|}$. See Figure 1.20.

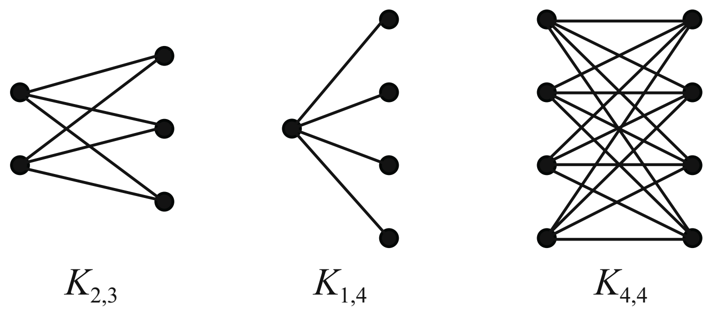


FIGURE 1.20. A few complete bipartite graphs.

Jump to the next page

The next theorem gives an interesting characterization of bipartite graphs.

Theorem 1.3. *A graph with at least two vertices is bipartite if and only if it contains no odd cycles.*

Proof. Let G be a bipartite graph with partite sets X and Y. Let C be a cycle of G and say that C is $v_1, v_2, \ldots, v_k, v_1$. Assume without loss of generality that $v_1 \in X$. The nature of bipartite graphs implies then that $v_i \in X$ for all odd i, and $v_i \in Y$ for all even i. Since v_k is adjacent to v_1, it must be that k is even; and hence C is an even cycle.

For the reverse direction of the theorem, let G be a graph of order at least two such that G contains no odd cycles. Without loss of generality, we can assume that G is connected, for if not, we could treat each of its connected components separately. Let v be a vertex of G, and define the set X to be

$$X = \{x \in V(G) \mid \text{the shortest path from } x \text{ to } v \text{ has even length}\},$$

and let $Y = V(G) \setminus X$.

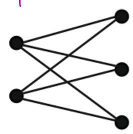
Now let x and x' be vertices of X, and suppose that x and x' are adjacent. If $x = v$, then the shortest path from v to x' has length one. But this implies that $x' \in Y$, a contradiction. So, it must be that $x \neq v$, and by a similar argument, $x' \neq v$. Let P_1 be a path from v to x of shortest length (a shortest v–x path) and let P_2 be a shortest v–x' path. Say that P_1 is $v = v_0, v_1, \ldots, v_{2k} = x$ and that P_2 is $v = w_0, w_1, \ldots, w_{2t} = x'$. The paths P_1 and P_2 certainly have v in common. Let v' be a vertex on both paths such that the v'–x path, call it P_1', and the v'–x' path, call it P_2', have only the vertex v' in common. Essentially, v' is the "last" vertex common to P_1 and P_2. It must be that P_1' and P_2' are shortest v'–x and v'–x' paths, respectively, and it must be that $v' = v_i = w_i$ for some i. But since x and x' are adjacent, $v_i, v_{i+1}, \ldots, v_{2k}, w_{2t}, w_{2t-1}, \ldots, w_i$ is a cycle of length $(2k - i) + (2t - i) + 1$, which is odd, and that is a contradiction.

Thus, no two vertices in X are adjacent to each other, and a similar argument shows that no two vertices in Y are adjacent to each other. Therefore, G is bipartite with partite sets X and Y. $\square$

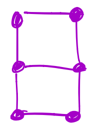
Theorem 1.3. *A* (connected) *graph with at least two vertices is bipartite if and only if it contains no odd cycles.* closed paths (p. 6)

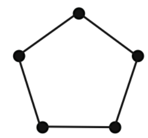
E.g. Both are bipartite:

Every cycle has length 4 or 6
(# of edges)



$K_{2,3}$





Not bipartite

Every cycle has length 5

Proof Let G be a graph with two or more vertices.

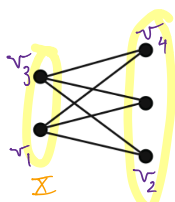
($\Rightarrow$)

First, show that if G is bipartite then G contains no odd cycles.
Suppose G is bipartite with partite sets X and Y.

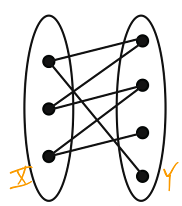
Let C be a cycle of G, say, C is $v_1, v_2, \dots, v_k, v_1$. (We will show C has even length.)
Assume without loss of generality that $v_1 \in X$.

Since G is bipartite, every edge of G has one end vertex in X and the other in Y, so $v_2 \in Y$, $v_3 \in X$, $v_4 \in Y$, and so on —
$v_i \in X$ iff i is odd, and $v_i \in Y$ iff i is even.

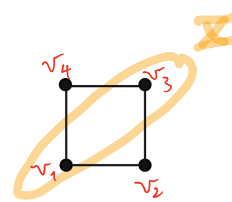
Since v_k is adjacent to $v_1 \in X$ and (again) G is bipartite, it must be that $v_k \in Y$, so k is even.
Hence C is an even cycle.



$K_{2,3}$



No cycle



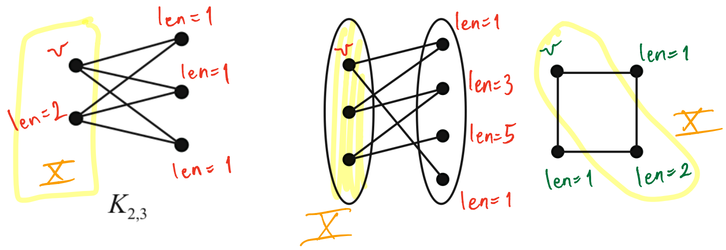
($\Leftarrow$)

For the reverse direction of the theorem, suppose G contains no odd cycles. (We will show that G is bipartite, i.e.
① we will partition the vertex set of G into two sets X and Y
② then show that every edge of G has one end vertex in X and the other in Y.)

① Pick a vertex v of G, and define the set X to be

$$X := \{ x \in \underbrace{V(G)}_{\text{vertices of } G} \mid \text{the shortest path from } x \text{ to } v \text{ has } \underbrace{\text{even length}}_{\text{even \# of edges}} \}$$

and let $Y := V(G) \setminus X$



Example 1 Example 2 Example 3

(2)

Claim 1: No two vertices in X are adjacent to each other.

Proof of Claim 1: Suppose (for contradiction) that x and x' are two adjacent vertices in X.

- If $x = v$, then the shortest path from v to x' has length one. This implies $x' \notin X$, contradicting the assumption that $x' \in X$.
- If $x' = v$, then the shortest path from v to x has length one. This implies $x \notin X$, contradicting the assumption that $x \in X$.
- So $x \neq v$ and $x' \neq v$.
- We will now construct a cycle in G.
- Let P_1 be a path from v to x of shortest length (a shortest v-x path). Say P_1 is $v = v_0, v_1, \dots, v_{2k-1}, v_{2k} = x$.
- Let P_2 be a path from v to x' of shortest length. (a shortest v-x' path). Say P_2 is $v = w_0, w_1, \dots, w_{2t-1}, w_{2t} = x'$.
- We know $k \geqslant 1, t \geqslant 1$ because $x \neq v$ and $x' \neq v$.

- The paths P_1 and P_2 certainly have v in common, since $v_0 = w_0$.

For example, $\begin{cases} P_1 \text{ may be } v = a, b, c, d, e = x \\ P_2 \text{ may be } v = \underset{v'}{a}, f, g = x' \end{cases}$

or $\begin{cases} P_1 \text{ may be } v = a, b, c, d, e = x \\ P_2 \text{ may be } v = a, b, \underset{v'}{c}, f, g = x' \end{cases}$

- Let v' be the "last" vertex common to P_1 and P_2. That is, let v' be a vertex on both paths P_1, P_2 such that the v'-x path (call it P_1') and the v'-x' path (call it P_2') have only the vertex v' in common.
- It must be that P_1' is a shortest $v'-x$ path
  and P_2' is a shortest $v'-x'$ path
  and $v' = v_i = w_i$ for some i.
- Then the vertices of P_1' and P_2' are distinct except for $v_i = w_i$, so $w_{2t}, w_{2t-1}, \dots, w_i = v' = v_i, v_{i+1}, \dots, v_{2k}$ is a path.

E.g. If $\begin{cases} P_1 \text{ may be } v=a,b,c,d,e=x \\ P_2 \text{ may be } v=\underset{v'}{a},f,g=x' \end{cases}$ then $x'=g,f,\underset{v'}{a},b,c,d,e=x$ is a path

If $\begin{cases} P_1 \text{ may be } v=a,b,c,d,e=x \\ P_2 \text{ may be } v=a,b,\underset{v'}{c},f,g=x' \end{cases}$ then $x'=g,f,c,d,e=x$ is a path

· Since $x=v_{2k}$ and $x'=w_{2t}$ are adjacent,

$x'=w_{2t}, w_{2t-1}, \ldots, w_i=v'=v_i, v_{i+1}, \ldots, v_{2k}=x, x'$

is a cycle (closed path) of length $(2k-i)+(2t-i)+1$, which is odd.

This contradicts the fact that G has no odd cycles.

Thus, no two vertices in X are adjacent to each other

[proof of claim 1]

<u>Claim 2</u> No two vertices in Y are adjacent to each other.

<u>Proof of Claim 2</u> Exercise.

By Claim 1 and Claim 2, every edge of G must have one end vertex in X and the other in Y. Therefore, G is bipartite (with partite sets X and Y.)

$\boxed{\text{Thm } 1.3}$

We conclude this section with a discussion of what it means for two graphs to be the same. Look closely at the graphs in Figure 1.21 and convince yourself that one could be re-drawn to look just like the other. Even though these graphs

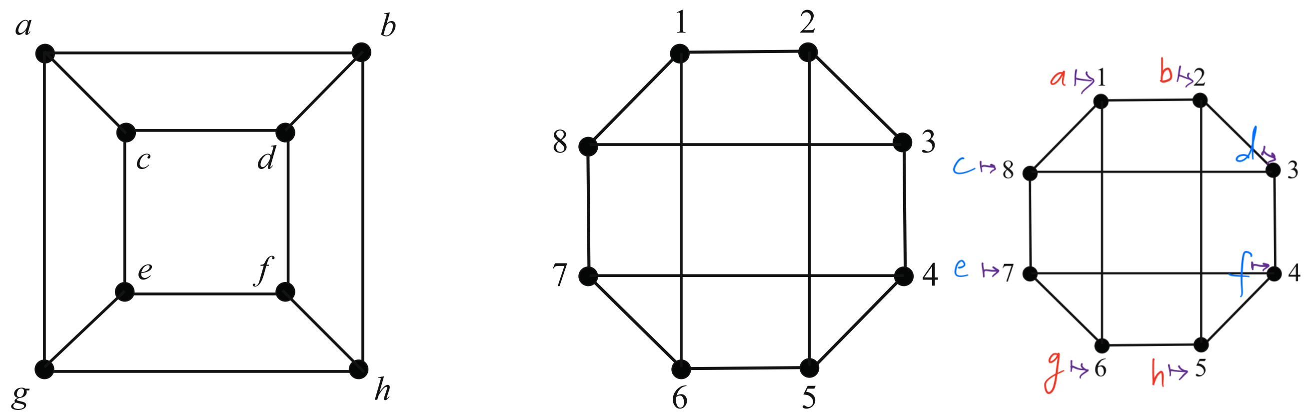


FIGURE 1.21. Are these graphs the same?

have different vertex sets and are drawn differently, it is still quite natural to think of these graphs as being the same. The idea of isomorphism formalizes this phenomenon.

Graphs G and H are said to be *isomorphic* to one another (or simply, isomorphic) if there exists a one-to-one correspondence $f : V(G) \to V(H)$ such that for each pair x,y of vertices of G, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. In other words, G and H are isomorphic if there exists a mapping from one vertex set to another that preserves adjacencies. The mapping itself is called an *isomorphism*. In our example, such an isomorphism could be described as follows:

$$\{(a,1),(b,2),(c,8),(d,3),(e,7),(f,4),(g,6),(h,5)\}\,.$$

When two graphs G and H are isomorphic, it is not uncommon to simply say that $G = H$ or that "G is H." As you will see, we will make use of this convention quite often in the sections that follow.

Several facts about isomorphic graphs are immediate. First, 1 if G and H are isomorphic, then $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$. The converse of this statement is not true, though, and you can see that in the graphs of Figure 1.22. The vertex and edge counts are the same, but the two graphs are clearly not iso-

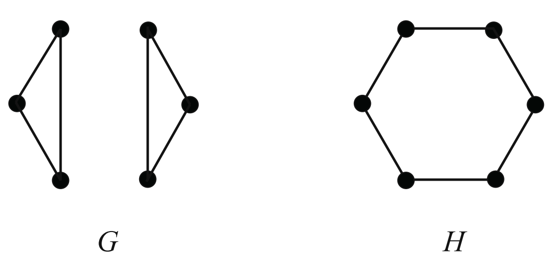


FIGURE 1.22.

morphic.

2

A second necessary fact is that if G and H are isomorphic then the degree sequences must be identical. Again, the graphs in Figure 1.22 show that the converse of this statement is not true. A third fact, and one that you will prove in Exercise 8, is that if graphs G and H are isomorphic, then their complements $\overline{G}$ and $\overline{H}$ must also be isomorphic.

In general, determining whether two graphs are isomorphic is a difficult problem. While the question is simple for small graphs and for pairs where the vertex counts, edge counts, or degree sequences differ, the general problem is often tricky to solve. A common strategy, and one you might find helpful in Exercises 9 and 10, is to compare subgraphs, complements, or the degrees of adjacent pairs of vertices.

Exercises

the complete graph on n vertices

1. For $n \geq 1$, prove that K_n has $n(n-1)/2$ edges.

K_{r_1,r_2} complete bipartite graph

2. If K_{r_1,r_2} is regular, prove that $r_1 = r_2$.

all vertices have the same deg.

3. Determine whether K_4 is a subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.
4. Determine whether P_4 is an induced subgraph of $K_{4,4}$. If yes, then exhibit it. If no, then explain why not.
5. List all of the unlabeled connected subgraphs of C_{34}.
6. The concept of complete bipartite graphs can be generalized to define the *complete multipartite graph* $K_{r_1,r_2,\ldots,r_k}$. This graph consists of k sets of vertices $A_1, A_2, \ldots, A_k$, with $|A_i| = r_i$ for each i, where all possible "interset edges" are present and no "intraset edges" are present. Find expressions for the order and size of $K_{r_1,r_2,\ldots,r_k}$.
7. The *line graph* $L(G)$ of a graph G is defined in the following way: the vertices of $L(G)$ are the edges of G, $V(L(G)) = E(G)$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.

   (a) Let G be the graph shown in Figure 1.23. Find $L(G)$.

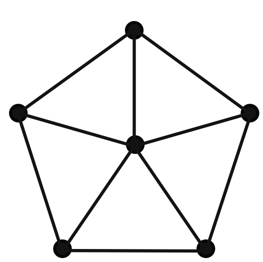


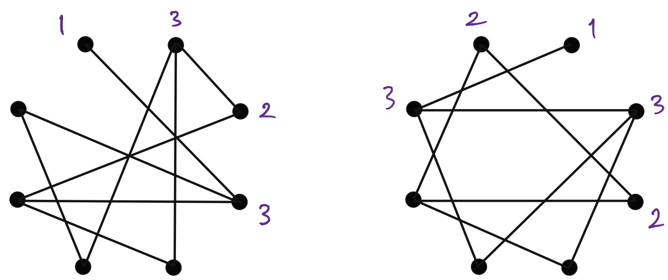
FIGURE 1.23.

(b) Find the complement of $L(K_5)$.

(c) Suppose G has n vertices, labeled $v_1, \ldots v_n$, and the degree of vertex v_i is r_i. Let m denote the size of G, so $r_1 + r_2 + \cdots + r_n = 2m$. Find formulas for the order and size of $L(G)$ in terms of n, m, and the r_i.

8. Prove that if graphs G and H are isomorphic, then their complements $\overline{G}$ and $\overline{H}$ are also isomorphic.

9. Prove that the two graphs in Figure 1.24 are not isomorphic. Check the degree sequence (p. 6) of each graph



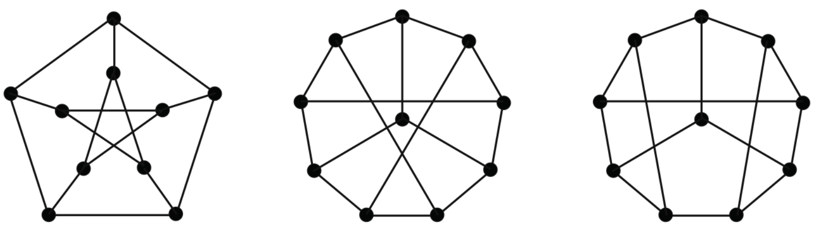
Every cycle is of length 4 or longer

FIGURE 1.24.

There is a cycle of length 3

10. Two of the graphs in Figure 1.25 are isomorphic.

Each is 3-regular (see p. 11)
vertices is 10.
So the sum of degrees is $10 \cdot 3 = 30$.
By the First Thm of Graph Theory, # edges is $\frac{30}{2} = 15$.



P *Q* *R*

Length of shortest cycles: ___

FIGURE 1.25.

Length of shortest cycles: ___

Length of shortest cycles: ___

(a) For the pair that is isomorphic, give an appropriate one-to-one correspondence.

(b) Prove that the remaining graph is not isomporhic to the other two.

1.2 Distance in Graphs

'Tis distance lends enchantment to the view . . .

— Thomas Campbell, *The Pleasures of Hope*

How far is it from one vertex to another? In this section we define distance in graphs, and we consider several properties, interpretations, and applications. Distance functions, called metrics, are used in many different areas of mathematics, and they have three defining properties. If M is a metric, then