1

Graph Theory

Textbook: "Combinatorics & Graph Theory"
by Harris, Hirst, Mossinghoff

"Begin at the beginning," the King said, gravely, "and go on till you come to the end; then stop."

— Lewis Carroll, *Alice in Wonderland*

The Pregolya River passes through a city once known as Königsberg. In the 1700s seven bridges were situated across this river in a manner similar to what you see in Figure 1.1. The city's residents enjoyed strolling on these bridges, but, as hard as they tried, no resident of the city was ever able to walk a route that crossed each of these bridges exactly once. The Swiss mathematician Leonhard Euler learned of this frustrating phenomenon, and in 1736 he wrote an article [98] about it. His work on the "Königsberg Bridge Problem" is considered by many to be the beginning of the field of graph theory.

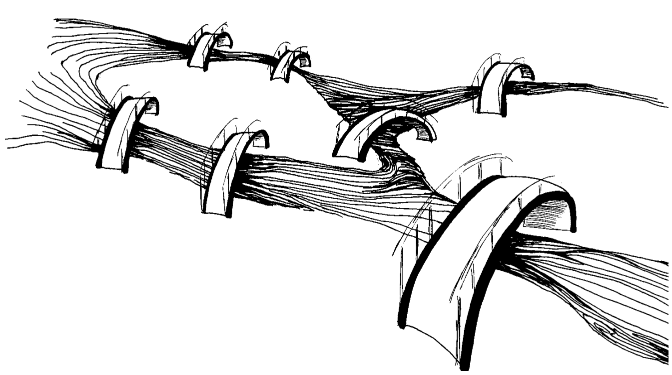


FIGURE 1.1. The bridges in Königsberg.

J.M. Harris et al., *Combinatorics and Graph Theory*, DOI: 10.1007/978-0-387-79711-3_1,

At first, the usefulness of Euler's ideas and of "graph theory" itself was found only in solving puzzles and in analyzing games and other recreations. In the mid 1800s, however, people began to realize that graphs could be used to model many things that were of interest in society. For instance, the "Four Color Map Conjecture," introduced by DeMorgan in 1852, was a famous problem that was seemingly unrelated to graph theory. The conjecture stated that four is the maximum number of colors required to color any map where bordering regions are colored differently. This conjecture can easily be phrased in terms of graph theory, and many researchers used this approach during the dozen decades that the problem remained unsolved.

The field of graph theory began to blossom in the twentieth century as more and more modeling possibilities were recognized — and the growth continues. It is interesting to note that as specific applications have increased in number and in scope, the theory itself has developed beautifully as well.

In Chapter 1 we investigate some of the major concepts and applications of graph theory. Keep your eyes open for the Königsberg Bridge Problem and the Four Color Problem, for we will encounter them along the way.

1.1 Introductory Concepts

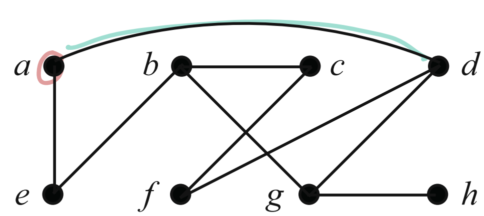
A definition is the enclosing a wilderness of idea within a wall of words.

— Samuel Butler, *Higgledy-Piggledy*

1.1.1 Graphs and Their Relatives

A *graph* consists of two finite sets, V and E. Each element of V is called a *vertex* (plural *vertices*). The elements of E, called *edges*, are unordered pairs of vertices. For instance, the set V might be $\{a, b, c, d, e, f, g, h\}$, and E might be $\{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\}$. Together, V and E are a graph G.

Graphs have natural visual representations. Look at the diagram in Figure 1.2. Notice that each element of V is represented by a small circle and that each element of E is represented by a line drawn between the corresponding two elements of V.



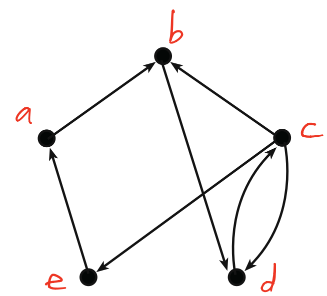
When we speak of a graph, think of this visual representation

FIGURE 1.2. A visual representation of the graph G.

As a matter of fact, we can just as easily define a graph to be a diagram consisting of small circles, called vertices, and curves, called edges, where each curve connects two of the circles together. When we speak of a graph in this chapter, we will almost always refer to such a diagram.

We can obtain similar structures by altering our definition in various ways. Here are some examples.

1. By replacing our set E with a set of *ordered* pairs of vertices, we obtain a *directed graph*, or *digraph* (Figure 1.3). Each edge of a digraph has a specific orientation.



$V =$

$E = \{ \{e,a\}, \{a,b\}, \{c,b\}, \quad , \quad , \quad , \quad \}$

FIGURE 1.3. A digraph.

2. If we allow repeated elements in our set of edges, technically replacing our set E with a multiset, we obtain a *multigraph* (Figure 1.4).

also called a graph

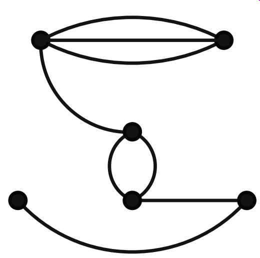


FIGURE 1.4. A multigraph.

3. By allowing edges to connect a vertex to itself ("loops"), we obtain a *pseudograph* (Figure 1.5).

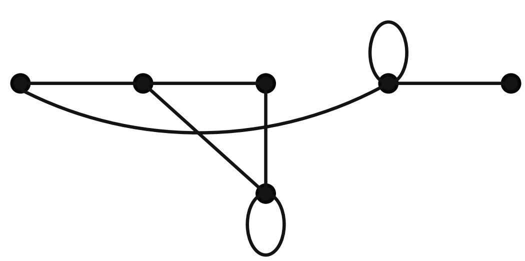


FIGURE 1.5. A pseudograph.

4. Allowing our edges to be arbitrary subsets of vertices (rather than just pairs) gives us *hypergraphs* (Figure 1.6).

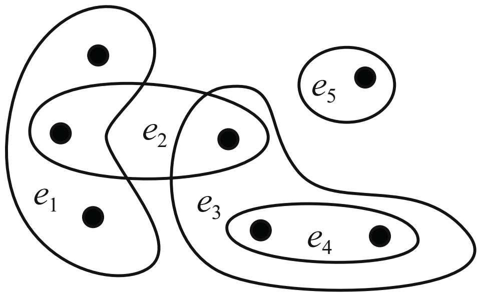
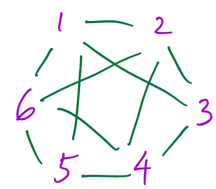


FIGURE 1.6. A hypergraph with 7 vertices and 5 edges.

5. By allowing V or E to be an infinite set, we obtain *infinite graphs*. Infinite graphs are studied in Chapter 3.

In this chapter we will focus on finite, simple graphs: those without loops or multiple edges.

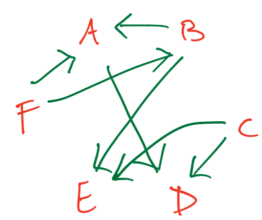
Exercises



1. ~~Ten~~ Six people are seated around a circular table. Each person shakes hands with everyone at the table except the person sitting directly across the table. Draw a graph that models this situation.

2. Six fraternity brothers (Adam, Bert, Chuck, Doug, Ernie, and Filthy Frank) need to pair off as roommates for the upcoming school year. Each person has compiled a list of the people with whom he would be willing to share a room.

   Adam's list: Doug
   Bert's list: Adam, Ernie
   Chuck's list: Doug, Ernie
   Doug's list: Chuck
   Ernie's list: Ernie
   Frank's list: Adam, Bert

   Draw a digraph that models this situation.

3. There are twelve women's basketball teams in the Atlantic Coast Conference: Boston College (B), Clemson (C), Duke (D), Florida State (F), Georgia Tech (G), Miami (I), NC State (S), Univ. of Maryland (M), Univ. of North Carolina (N), Univ. of Virginia (V), Virginia Tech (T), and Wake Forest Univ. (W). At a certain point in midseason,

   B has played I, T*, W

   C has played D*, G

D has played C*, S, W

F has played N*, V

G has played C, M

I has played B, M, T

S has played D, V*

M has played G, I, N

N has played F*, M, W

V has played F, S*

T has played B*, I

W has played B, D, N

The asterisk(*) indicates that these teams have played each other twice. Draw a multigraph that models this situation.

4. Can you explain why no resident of Königsberg was ever able to walk a route that crossed each bridge exactly once? (We will encounter this question again in Section 1.4.1.)

1.1.2 The Basics

Your first discipline is your vocabulary;

— Robert Frost

In this section we will introduce a number of basic graph theory terms and concepts. Study them carefully and pay special attention to the examples that are provided. Our work together in the sections that follow will be enriched by a solid understanding of these ideas.

The Very Basics

The vertex set of a graph G is denoted by $V(G)$, and the edge set is denoted by $E(G)$. We may refer to these sets simply as V and E if the context makes the particular graph clear. For notational convenience, instead of representing an edge as $\{u, v\}$, we denote this simply by uv. The *order* of a graph G is the cardinality of its vertex set, and the *size* of a graph is the cardinality of its edge set.

Given two vertices u and v, if $uv \in E$, then u and v are said to be *adjacent*. In this case, u and v are said to be the *end vertices* of the edge uv. If $uv \notin E$, then u and v are *nonadjacent*. Furthermore, if an edge e has a vertex v as an end vertex, we say that v is *incident* with e.

The *neighborhood* (or *open neighborhood*) of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v:

$$N(v) = \{x \in V \mid vx \in E\}.$$

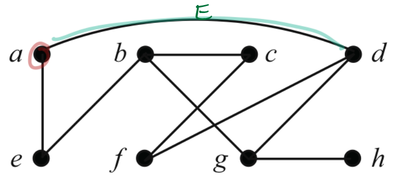


FIGURE 1.2. A visual representation of the graph G.

The order of G is $|\{a, b, c, \ldots, h\}| = 8$

The degree sequence of G is
3, 3, 3, 2, 2, 2, 2, 1

Ex. vertices a and d are adjacent
vertices a and b are nonadjacent
vertex a is incident with edge E,
and so is vertex d.

The *closed neighborhood* of a vertex v, denoted by $N[v]$, is simply the set $\{v\} \cup N(v)$. Given a set S of vertices, we define the neighborhood of S, denoted by $N(S)$, to be the union of the neighborhoods of the vertices in S. Similarly, the closed neighborhood of S, denoted $N[S]$, is defined to be $S \cup N(S)$.

The *degree* of v, denoted by $\deg(v)$, is the number of edges incident with v. In simple graphs, this is the same as the cardinality of the (open) neighborhood of v. The *maximum degree* of a graph G, denoted by $\Delta(G)$, is defined to be

$$\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}.$$

Similarly, the *minimum degree* of a graph G, denoted by $\delta(G)$, is defined to be

$$\delta(G) = \min\{\deg(v) \mid v \in V(G)\}.$$

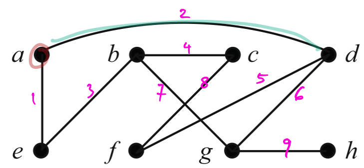
The *degree sequence* of a graph of order n is the n-term sequence (usually written in descending order) of the vertex degrees.

Let's use the graph G in Figure 1.2 to illustrate some of these concepts: G has order 8 and size 9; vertices a and e are adjacent while vertices a and b are nonadjacent; $N(d) = \{a, f, g\}$, $N[d] = \{a, d, f, g\}$; $\Delta(G) = 3$, $\delta(G) = 1$; and the degree sequence is 3, 3, 3, 2, 2, 2, 2, 1.

The following theorem is often referred to as the First Theorem of Graph Theory.

Theorem 1.1. (1) *In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges.* (2) *Consequently, the number of vertices with odd degree is even.*

Ex

$G =$ 

$\deg(a) = 2$
$\deg(b) = 3$
$\deg(c) = 2$
$\deg(d) = 3$
$\deg(e) = 2$
$\deg(f) = 2$
$\deg(g) = 3$
$\deg(h) = 1$ +

sum of degrees 18

$|E(G)| = 9$

$\{\text{vertices w/ odd degree}\} = \{b, d, g, h\}$

since each edge has two endpoints

Proof. (1) Let $S = \sum_{v \in V} \deg(v)$. Notice that in counting S, we count each edge exactly twice. Thus, $S \overset{(*)}{=} 2|E|$ (the sum of the degrees is twice the number of edges). (2) Since S is even, it must be that the number of vertices with odd degree is even. To prove this claim, suppose otherwise.

Let V_o be the set of vertices with odd degree and V_e the set of vertices with even degree.

Then $\sum_{v \in V_o} \deg(v)$ is odd because $|V_o|$ is odd and each $\deg(v)$ is odd for $v \in V_o$,

and $\sum_{v \in V_e} \deg(v)$ is even because it's the sum of even numbers.

So $S = \sum_{v \in V_o} \deg(v) + \sum_{v \in V_e} \deg(v)$ is odd, which contradicts $(*)$. So $|V_o|$ must be even $\square$

Perambulation and Connectivity

A *walk* in a graph is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, k-1$. Such a walk is sometimes called a v_1–v_k *walk*, and v_1 and v_k are the *end vertices* of the walk. If the vertices in a walk are distinct, then the walk is called a *path*. If the edges in a walk are distinct, then the walk is called a *trail*. In this way, every path is a trail, but not every trail is a path. Got it?

A *closed path*, or *cycle*, is a path $v_1, \ldots, v_k$ (where $k \geq 3$) together with the edge $v_k v_1$. Similarly, a trail that begins and ends at the same vertex is called a *closed trail*, or *circuit*. The *length* of a walk (or path, or trail, or cycle, or circuit) is its number of edges, counting repetitions.

Once again, let's illustrate these definitions with an example. In the graph of Figure 1.7, a, c, f, c, b, d is a walk of length 5. The sequence b, a, c, b, d represents a trail of length 4, and the sequence d, g, b, a, c, f, e represents a path of length 6.

A walk of length 5 not a trail not a path

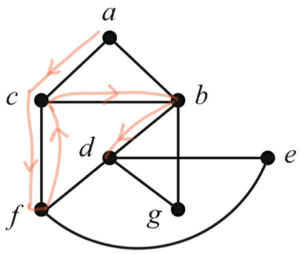


FIGURE 1.7.

(Also a walk) A trail of length 4 not a path

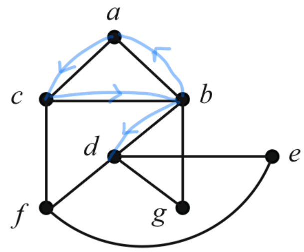


FIGURE 1.7.

Also a walk and a trail A path of length 6

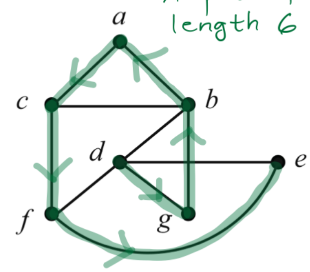


FIGURE 1.7.

Also, g, d, b, c, a, b, g is a circuit, while e, d, b, a, c, f, e is a cycle. In general, it is possible for a walk, trail, or path to have length 0, but the least possible length of a circuit or cycle is 3. (for a simple graph)

a single vertex

The following theorem is often referred to as the Second Theorem in this book.

may repeat vertices, edges

Theorem 1.2. *In a graph G with vertices u and v, every u–v walk contains a u–v path* (i.e a walk with distinct vertices). simple

Proof. Let W be a u–v walk in G. We prove this theorem by induction on the length of W. If W is of length 1 or 2, then it is easy to see that W must be a path.

$v_1 \to v_2$ length 1

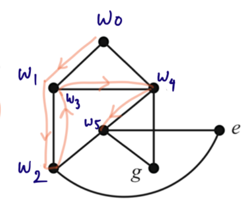
$v_1 \to v_2 \to v_3$ length 2

$v_1 \rightleftarrows v_2$ Not possible because G is simple

For the induction hypothesis, suppose the result is true for all walks of length less than k, and suppose W has length k. Say that W is

$$u = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v$$

W

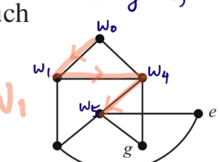


where the vertices are not necessarily distinct. If the vertices are in fact distinct, then W itself is the desired u–v path. If not, then let j be the smallest integer such that $w_j = w_r$ for some $r > j$. Let W_1 be the walk

Here $j = 1$, $r = 3$

$$u = w_0, \ldots, w_j, w_{r+1}, \ldots, w_k = v.$$

W_1



This walk has length strictly less than k, and therefore the induction hypothesis implies that W_1 contains a u–v path. This means that W contains a u–v path, and the proof is complete. □

We now introduce two different operations on graphs: *vertex deletion* and *edge deletion*. Given a graph G and a vertex $v \in V(G)$, we let $G - v$ denote the graph obtained by removing v and all edges incident with v from G. If S is a set of vertices, we let $G - S$ denote the graph obtained by removing each vertex of S and all associated incident edges. If e is an edge of G, then $G - e$ is the graph obtained by removing only the edge e (its end vertices stay). If T is a set of edges, then $G - T$ is the graph obtained by deleting each edge of T from G. Figure 1.8 gives examples of these operations.

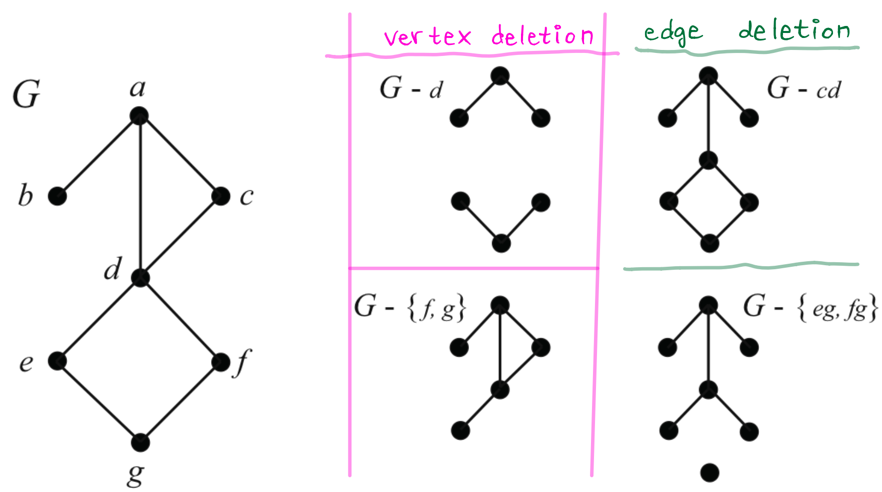


FIGURE 1.8. Deletion operations.

A graph is *connected* if every pair of vertices can be joined by a path. Informally, if one can pick up an entire graph by grabbing just one vertex, then the graph is connected. In Figure 1.9, G_1 is connected, and both G_2 and G_3 are not connected (or *disconnected*). Each maximal connected piece of a graph is called a *connected component*. In Figure 1.9, G_1 has one component, G_2 has three components, and G_3 has two components.

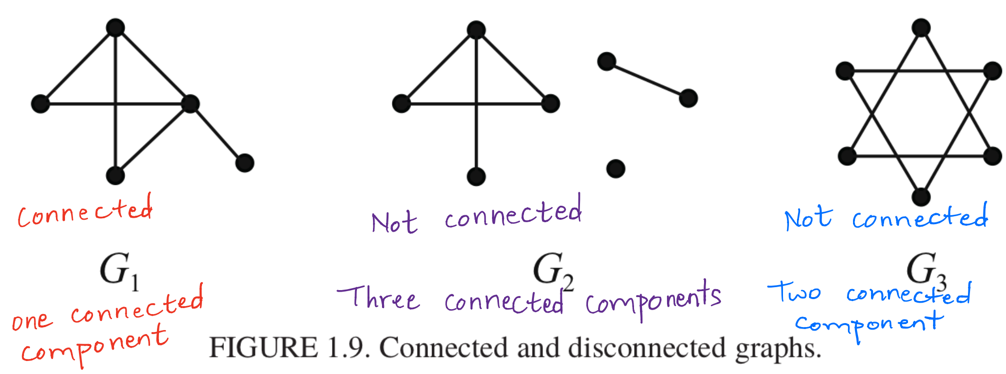


FIGURE 1.9. Connected and disconnected graphs.

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If the deletion of a vertex v from G causes the number of components to increase, then v is called a *cut vertex*. In the graph G of Figure 1.8, vertex d is a cut vertex and vertex c is not. Similarly, an edge e in G is said to be a *bridge* if the graph $G - e$ has more components than G. In Figure 1.8, the edge ab is the only bridge.

A proper subset S of vertices of a graph G is called a *vertex cut set* (or simply, a *cut set*) if the graph $G - S$ is disconnected. A graph is said to be *complete* if every vertex is adjacent to every other vertex. Consequently, if a graph contains at least one nonadjacent pair of vertices, then that graph is not complete. Complete graphs do not have any cut sets, since $G - S$ is connected for all proper subsets S of the vertex set. Every non-complete graph has a cut set, though, and this leads us to another definition. For a graph G which is not complete, the *connectivity* of G, denoted $\kappa(G)$, is the minimum size of a cut set of G. If G is a connected, non-complete graph of order n, then $1 \leq \kappa(G) \leq n - 2$. If G is disconnected, then $\kappa(G) = 0$. If G is complete of order n, then we say that $\kappa(G) = n - 1$.

Further, for a positive integer k, we say that a graph is *k-connected* if $k \leq \kappa(G)$. You will note here that "1-connected" simply means "connected."

Here are several facts that follow from these definitions. You will get to prove a couple of them in the exercises.

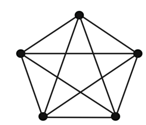
i. A graph is connected if and only if $\kappa(G) \geq 1$.

ii. $\kappa(G) \geq 2$ if and only if G is connected and has no cut vertices.

iii. Every 2-connected graph contains at least one cycle.

iv. For every graph G, $\kappa(G) \leq \delta(G)$.

E.g. For $n=5$, the answer is the # of edges of



Sec 1.1.2 **Exercises**

G has n vertices

1. If G is a graph of order n, what is the maximum number of edges in G?

2. Prove that for any graph G of order at least 2, the degree sequence has at least one pair of repeated entries. Proof Let $n \geq 2$. Suppose the degree sequence has no repeated entries. Then ...

3. Consider the graph shown in Figure 1.10.

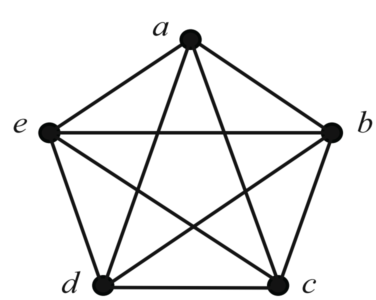


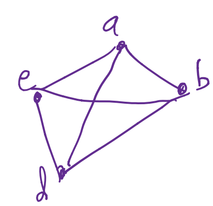
FIGURE 1.10.

Count by length

length	# paths
0	1
1	# vertices except c
2	4 . 3
3	
4	

   (a) How many different paths have c as an end vertex?

   (b) How many different paths avoid vertex c altogether? (Consider $G-c$)



   (c) What is the maximum length of a circuit in this graph? Give an example of such a circuit. closed trail, distinct edges

   (d) What is the maximum length of a circuit that does not include vertex c? Give an example of such a circuit.

4. Is it true that a finite graph having exactly two vertices of odd degree must contain a path from one to the other? Give a proof or a counterexample.

5. Let G be a graph where $\delta(G) \geq k$.

   (a) Prove that G has a path of length at least k.

   (b) If $k \geq 2$, prove that G has a cycle of length at least $k+1$.

6. Prove that every closed odd walk in a graph contains an odd cycle.

7. Draw a connected graph having at most 10 vertices that has at least one cycle of each length from 5 through 9, but has no cycles of any other length.

8. Let P_1 and P_2 be two paths of maximum length in a connected graph G. Prove that P_1 and P_2 have a common vertex.

9. Let G be a graph of order n that is not connected. What is the maximum size of G?

10. Let G be a graph of order n and size strictly less than $n-1$. Prove that G is not connected.

11. Prove that an edge e is a bridge of G if and only if e lies on no cycle of G.

12. Prove or disprove each of the following statements.

    (a) If G has no bridges, then G has exactly one cycle.
    (b) If G has no cut vertices, then G has no bridges.
    (c) If G has no bridges, then G has no cut vertices.

13. Prove or disprove: If every vertex of a connected graph G lies on at least one cycle, then G is 2-connected.

14. Prove that every 2-connected graph contains at least one cycle.

15. Prove that for every graph G,

    (a) $\kappa(G) \leq \delta(G)$;
    (b) if $\delta(G) \geq n-2$, then $\kappa(G) = \delta(G)$.

16. Let G be a graph of order n.

    (a) If $\delta(G) \geq \frac{n-1}{2}$, then prove that G is connected.
    (b) If $\delta(G) \geq \frac{n-2}{2}$, then show that G need not be connected.

1.1.3 Special Types of Graphs

until we meet again . . .

— from *An Irish Blessing*

In this section we describe several types of graphs. We will run into many of them later in the chapter.

1. Complete Graphs

   We introduced complete graphs in the previous section. A complete graph of order n is denoted by K_n, and there are several examples in Figure 1.11.