Math3250 Combinatorics Week6 Problems (Study Prep for Exam 1)

1 Nine vectors

Let $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \ldots, \begin{pmatrix} a_9 \\ b_9 \\ c_9 \end{pmatrix}$ be nine vectors in $\mathbb{Z}^3$. Prove that at least two of these nine vectors have a sum whose coordinates are all even integers.

Solution:

Proof. Given an integer x, let $x' = 0$ if x is even and 1 if x is odd. There are eight possible sequences of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$; hence, by the pigeonhole-principle, there two distinct $i, j \in [9]$ such that $\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix}$. Then the sum $\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} + \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} = \begin{pmatrix} a_i + a_j \\ b_j + b_j \\ c_i + c_j \end{pmatrix}$ of these two vectors have even coordinates. □

2 RIFFRAFF

How many different ways are there to arrange the letters in the word RIFFRAFF? How many different ways are there to arrange the letters in the word RIFFRAFF so that the two R's are not adjacent?

Solution:

Answer. The number of all arrangements of the multiset of 8 objects with 4 objects (that look like F), 2 objects (that look like R), one I, and 1 A is $\binom{8}{4,2,1,1}$. The number of arrangements where the two R's *are* adjacent is the same as the number of all arrangements of the multiset of 7 objects with 4 objects (that look like F), 1 object (that looks like RR), one I, and 1 A. Subtracting the second number from the first number we get $\boxed{\binom{8}{4,2,1,1} - \binom{7}{4,1,1,1}} = 4 \cdot 7 \cdot 6 \cdot 5 - \cdot 7 \cdot 6 \cdot 5 = 3 \cdot 7 \cdot 6 \cdot 5 = \boxed{630}$. □

3 Binary words

Let $f(k)$ be the number of binary sequences $a_1, a_2, \ldots, a_k$ (note that this means that each a_i is 0 or 1). Note that $f(0) = 1$ because there is one binary sequence of length 0, empty sequence. Find a simple formula for $f(n)$.

Solution:

Answer. This is a special case of Theorem 3.6 in Bona, where $n = 2$ (see the proof in the book on page 46-47).

For each a_i, there are two options, so $\boxed{f(n) = 2^k}$. Read the first theorem of Section 3.2 in Bona and its proof. □

4 Bijections

Let $n \geq 4$. How many bijections $\pi : [n] \to [n]$ satisfy $\pi(1) = 2$, $\pi(2) \neq 3$, $\pi(2) \neq 4$, and $\pi(3) \neq 4$? Give a simple formula not involving summation symbols.

(Afterwards, you should check that your formula works for $n = 4$).

Solution:

Answer. There is only one choice for $\pi(1)$. There are then $n-3$ choices for $\pi(2)$ (anything other than 2, 3, and 4). There are then $n-3$ choices for $\pi(3)$ (anything other than 2, $\pi(2)$, and 4, which are all different). There are then $n-3$ choices for $\pi(4)$, $n-4$ choices for $\pi(5)$, etc. This gives $\boxed{(n-3)^3(n-4)! = (n-3)^2(n-3)!}$ choices in all. (For $n=4$, the formula gives 1, and the only such bijection is the map sending 1 to 2, 2 to 1, and fixing both 3 and 4 pointwise.) □

5 Finding an identity

Find a simple formula (no summation symbols) for

$$f(n) = \sum_{k=0}^{n} \binom{k}{2}\binom{n}{k}.$$

Solution:

Answer 1. The right-hand side counts the number of ways to choose a subset S of any size from $[n]$, then choose a 2-element subset T from S. But we could get the same result by choosing T first in $\binom{n}{2}$ ways, then choose an arbitrary subset of the remaining $n-2$ elements in 2^{n-2} ways, which gives $f(n) = \binom{n}{2}2^{n-2}$. □

Answer 2. Take the binomial expansion $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, differentiate twice and then divide both sides by 2. Then set $x=1$ to get

$$\binom{n}{2}2^{n-2} = \sum_{k=0}^{n}\binom{k}{2}\binom{n}{k}$$

□

6 Enumerating all subsets

Given a positive integer n, what is the the number of all subsets of $[n]$?

a. Prove by induction on n.

Solution: The answer is 2^n.

Proof (by induction) from Theorem 2.4 page 27 of Bona. For $n=1$, the statement is true as $[1] = \{1\}$ has two subsets, the empty set, and $\{1\}$.

Now let k be a positive integer, and assume that the statement is true for $n=k$. We divide the subset of $[k+1]$ into two classes: there will be those subsets that do not contain the element $k+1$, and there will be those that do. Those that do not contain $k+1$ are also subsets of $[k]$, so by the induction hypothesis their number is 2^k. Those that contain $k+1$ consist of $k+1$ and a subset of $[k]$. However, that subset of $[k]$ can be any of the 2^k subsets of $[k]$, so the number of these subsets of $[k+1]$ is once more 2^k. So altogether, $[k+1]$ has $2^k + 2^k = 2^{k+1}$ subsets, and the statement is proven. □

b. Prove by another method.

Solution: There is a proof using a bijection in Section 3.2 and there is a proof in Section 4.1 using the binomial theorem.

7 A sequence

Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and let

$$a_{n+1} = 2(a_0 + a_1 + \cdots + a_n)$$

for $n \geq 0$. Conjecture an explicit formula for a_n for $n \geq 1$, and prove it.

Solution: Prove that $a_n = 2 \cdot 3^{n-1}$ for $n \geq 1$.

Proof. We prove this by strong induction on n. Since $2(a_0) = 2(1) = 2 \cdot 3^{1-1}$, the initial case (for $n = 1$) is verified. Now let us assume that the statement is true for all positive integers that are less than or equal to n. Then, we have

$$\begin{aligned} a_{n+1} &= 2(a_0 + a_1 + a_2 + \cdots + a_n) \quad \text{by the recurrence relation} \\ &= 2a_0 + 2(a_1 + a_2 + \cdots + a_n) \\ &= 2 + 2(2 \cdot 1 + 2 \cdot 3 + \cdots + 2 \cdot 3^{n-1}) \quad \text{by the induction hypothesis} \\ &= 2 + 4(1 + 3 + \cdots + 3^{n-1}) \\ &= 2 + 4\left(\frac{3^n - 1}{2}\right) \quad \text{since the series is a geometric series} \\ &= 2 + 2(3^n - 1) \\ &= 2 \cdot 3^n. \end{aligned}$$

This proves that our explicit formula is correct for $n + 1$, and the proof is complete. □

8 Integrate

Find an explicit (closed-form expression) formula for the expression

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} 5^{k+1}.$$

Solution: (Review Section 4.1 binomial theorem.) Expand $(1 + x)^n$ using the binomial theorem, so that you have an equation with $(1 + x)^n$ on the right-hand side and the expansion on the left-hand side. This function is a polynomial, so we can integrate both sides of the equation to get

$$\sum_{k=0}^{n} \binom{n}{k} x^{k+1} \frac{x^{k+1}}{k+1} + C = \frac{(1+x)^{n+1}}{n+1}$$

To solve for the constant C, set x to a convenient value. In this case, it's convenient to set x to 0, and we get

$$C = \frac{1}{n+1}.$$

Hence

$$\sum_{k=0}^{n} \binom{n}{k} x^{k+1} \frac{x^{k+1}}{k+1} + \frac{1}{n+1} = \frac{(1+x)^n}{n+1}. \tag{1}$$

To solve the original problem, set $x = 5$ to equation (1), so

$$\sum_{k=0}^{n} \binom{n}{k} x^{k+1} \frac{5^{k+1}}{k+1} = \boxed{\frac{6^{n+1}}{n+1} - \frac{1}{n+1}}$$

9 Seating

Suppose you have 7 people which you wish to seat at four *ordered* tables, labeled 1, 2, 3, and 4. You do not care how people are seated within each table. From those people sitting in table 1, you will choose one person to be the president. Everyone must be seated, but it's OK if some tables are empty (except that table 1 must has at least one person, since the president sits in table 1).

How many different ways can we do this?

Solution: The answer is $7 \cdot 4^6$.

Proof. The number of ways to choose a president is 7. After choosing the president, we can assign a table for each of the remaining 6 people. Since there are four tables to choose from, there are 4^6 ways to do this. $\square$

Note: This is the same number as the number of ways to give each of the 7 of you a shirt (with four possible colors, red, white, blue, purple) and choose one scribe which must wear the red shirt.

10 ABCD Identity

Prove the identity

$$\sum_{a+b+c+d=n} a\binom{n}{a,b,c,d} = n\ 4^{n-1}$$

where the sum is taken over all tuples (a, b, c, d) of nonnegative integers satisfying $a + b + c + d = n$.

Give a combinatorial proof and a proof using the multinomial theorem.

Solution: Note: This is the same counting problem as Problem 9.

Combinatorial Proof. Both sides count the number of ways to do the following. We have n people which have to sit at four *ordered* tables, labeled 1, 2, 3, and 4. We do not care how people are seated within each table. From those people sitting in table 1, we will choose one person to be the president. Everyone must be seated, but it's OK if some tables are empty (except that table 1 must has at least one person, since the president sits in table 1).

On the right-hand side, we choose the president (out of n possibilities), then choose the table assignment (four options for each of the remaining $n - 1$ people).

We want to show that the left-hand side counts the same thing. First, note that, if a, b, c, d are fixed, then $\binom{n}{a,b,c,d}$ is the number of ways to assign a people to the first table, b people to the second table, c people to third table, and d people to the fourth table. (To see this, imagine lining up the n people in one row in $n!$ ways, then moving the first a people to the first table, the next b people to the second table, and so on. Since we don't care how people are seated within each table, we divide $n!$ by $a!b!c!d!$.)

On the left-hand side, we choose the table assignment first (separated into cases for each tuple (a, b, c, d)), then we choose the president (out of the a people sitting in table 1). $\square$

Solution:

Proof using the Multinomial Theorem. The Multinomial Theorem gives

$$\sum_{a+b+c+d=n} \binom{n}{a,b,c,d} x^a y^b z^c w^d = (x + y + z + w)^n$$

where the sum is taken over nonnegative integers a, b, c, d satisfying $a + b + c + d = n$. Polynomials are differentiable, so we can differentiate both sides with respect to x, and get

$$\sum_{a+b+c+d=n} a\binom{n}{a,b,c,d} x^{a-1} y^b z^c w^d = n(x + y + z + w)^{n-1}$$

where the sum is taken over nonnegative integers a, b, c, d satisfying $a + b + c + d = n$ and $a > 0$. Now set $x = y = z = w = 1$.

$\square$

11 Expansion

Use the binomial theorem and the methods shown in Section 4.3.

a. Compute the power series expansion of $\frac{1}{\sqrt{1-4x}}$.

Solution: See the solution of Chapter 4 Exercise 27 on page 98.

b. Compute the power series expansion of $\sqrt{1-4x}$.

Solution: See the solution of Chapter 4 Example 4.16 on page 82–83.

12 Restaurant with eight people

You and seven of your friends are going to a new restaurant at Storrs Center.

You decide to order food to share as a group. Half of the group is vegan and half is not. You decide to pick a committee of four people who will choose the food to share. *One of the committee members will be president.* To make sure the vegan diners will have some good options, everyone agrees the president will be vegan. How many ways can you choose this 4-people committee and a president?

Solution:

We pick a president (out of 4 vegans), then pick 3 people (out of the 8-1=7 remaining non-presidents).

$$\boxed{4\binom{7}{3} = \frac{4(7!)}{3!4!}}$$

13 Find a counting problem

Come up with a counting problem which solution is

$$n\binom{2n-1}{n-1}.$$

Solution:

Counting problem. A group of $2n$ friends are going to a new restaurant at Storrs Center. Half of the group is vegan and half is not. They pick a committee of n people who will choose the food to share. *One of the committee members will be president.* To make sure the vegan diners will have some good options, everyone agrees the president will be vegan. How many ways can you choose this n-people committee and a president? $\square$

Proof. We first choose a vegan president in n ways, then choose $(n-1)$ more committee members (out of the possible $2n-1$ non-president diners) in $\binom{2n-1}{n-1}$ ways.

So the number of ways to do this is

$$n\binom{2n-1}{n-1}.$$

□

14 Practice problems and solutions from the book

- All exercises (that has solution) at the end of Chapter 1 (Piegon Hole Principle) except the ones with (+) or (++) or (+++) symbols.
- All exercises (that has solution) at the end of Chapter 2 (induction) except the ones with (+) or (++) symbols.
- All exercises (that has solution) at the end of Chapter 3 (Elementary Counting problems) except the ones with (+) symbols.
- All exercises (that has solution) at the end of Chapter 4 (Binomial Theorem and Related Identities) except the ones with (+) symbols.