Math3250 Combinatorics Week6 Problems (Study Prep for Exam 1)

1   Nine vectors

Let $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$, $\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \ldots$, $\begin{pmatrix} a_9 \\ b_9 \\ c_9 \end{pmatrix}$ be nine vectors in $\mathbb{Z}^3$. Prove that at least two of these nine vectors have a sum whose coordinates are all even integers.

2   RIFFRAFF

How many different ways are there to arrange the letters in the word RIFFRAFF? How many different ways are there to arrange the letters in the word RIFFRAFF so that the two R's are not adjacent?

3   Binary words

Let $f(n)$ be the number of binary sequences a_1, a_2, ..., a_n (note that this means that each a_i is 0 or 1). Note that $f(0) = 1$ because there is one binary sequence of length 0, empty sequence. Find a simple formula for $f(n)$.

4   Bijections

Let $n \geq 4$. How many bijections $\pi : [n] \to [n]$ satisfy $\pi(1) = 2$, $\pi(2) \neq 3$, $\pi(2) \neq 4$, and $\pi(3) \neq 4$? Give a simple formula not involving summation symbols.

   (Afterwards, you should check that your formula works for $n = 4$).

5   Finding an identity

Find a simple formula (no summation symbols) for

$$f(n) = \sum_{k=0}^{n} \binom{k}{2} \binom{n}{k}.$$

6   Enumerating all subsets

Given a positive integer n, what is the the number of all subsets of $[n]$?

a. Prove by induction on n.

b. Prove by another method.

7   A sequence

Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and let

$$a_{n+1} = 2(a_0 + a_1 + \cdots + a_n)$$

for $n \geq 0$. Conjecture an explicit formula for a_n for $n \geq 1$, and prove it.

8   Integrate

Find an explicit (closed-form expression) formula for the expression

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} 5^{k+1}.$$

9 Seating

Suppose you have 7 people which you wish to seat at four *ordered* tables, labeled 1, 2, 3, and 4. You do not care how people are seated within each table. From those people sitting in table 1, you will choose one person to be the president. Everyone must be seated, but it's OK if some tables are empty (except that table 1 must has at least one person, since the president sits in table 1).

How many different ways can we do this?

10 ABCD Identity

Prove the identity

$$\sum_{a+b+c+d=n} a \binom{n}{a,b,c,d} = n \, 4^{n-1}$$

where the sum is taken over all tuples (a, b, c, d) of nonnegative integers satisfying $a + b + c + d = n$.

Give a combinatorial proof and a proof using the multinomial theorem.

11 Expansion

Use the binomial theorem and the methods shown in Section 4.3.

a. Compute the power series expansion of $\frac{1}{\sqrt{1-4x}}$.

b. Compute the power series expansion of $\sqrt{1 - 4x}$.

12 Restaurant with eight people

You and seven of your friends are going to a new restaurant at Storrs Center.

You decide to order food to share as a group. Half of the group is vegan and half is not. You decide to pick a committee of four people who will choose the food to share. *One of the committee members will be president.* To make sure the vegan diners will have some good options, everyone agrees the president will be vegan. How many ways can you choose this 4-people committee and a president?

13 Find a counting problem

Come up with a counting problem which solution is

$$n \binom{2n - 1}{n - 1}.$$

14 Practice problems and solutions from the book

- All exercises (that has solution) at the end of Chapter 1 (Piegon Hole Principle) except the ones with (+) or (++) or (+++) symbols.

- All exercises (that has solution) at the end of Chapter 2 (induction) except the ones with (+) or (++) symbols.

- All exercises (that has solution) at the end of Chapter 3 (Elementary Counting problems) except the ones with (+) symbols.

- All exercises (that has solution) at the end of Chapter 4 (Binomial Theorem and Related Identities) except the ones with (+) symbols.