Math3250 Combinatorics Problems Week 5

For the sake of learning the tools of Section 4.1, when you work on Problems 1 - 6 below, please model your explanations after the the proofs for Theorems 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7 from Section 4.1.

1. Sums

Let n be a positive integer. Prove that the identities

$$\sum_{k=0}^{n} 2^k \binom{n}{k} (-1)^{n-k} = 1 \qquad \text{and} \qquad \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

hold.
Hint: Read the proofs of Thm 4.2 and Thm 4.4 on page 74. A similar idea should work.

2. Inequality

Let n be a positive integer larger than 1. Prove that

$$2^n < \binom{2n}{n} < 4^n.$$

Hints:

- What objects are counted by $\binom{2n}{n}$? See the definition in Sec 3.3 Choice Problems and also the proofs given in Sec 4.1.
- What objects are counted by 2^n? See Example 3.11 in Sec 3.2.
- What objects are counted by 4^n? Would it be helpful to write 4^n as 2^{2n}?
- For inspirations, please read the combinatorial proofs given for the identities in Sec 4.1 for Theorem 4.3, Theorem 4.4, Theorem 4.5, and Theorem 4.6.

(Optional) Attempt to prove a generalization of above: Let k be an integer where $2 \leq k < n$. Show that the inequality

$$k^n < \binom{kn}{n} \text{ holds.}$$

3. An identity

Read the proofs of Theorem 4.6 on page 76. Then use a similar method to the first proof (the combinatorial proof) to show that

$$\sum_{j=2}^{n} j(j-1)\binom{n}{j} = n(n-1)2^{n-2}.$$

4. A more challenging identity

Let $n \geq 2$ be an integer. Show that

$$\sum_{j=1}^{n} j^2 \binom{n}{j} = n(n+1)2^{n-2}.$$

Give a combinatorial proof and also a proof using the Binomial Theorem (see Thm 4.6).

5. Yet another identity

Let k and n be positive integers such that $k < n$. Show that

$$\sum_{j=k}^{n} \binom{j}{k}\binom{n}{j} = \binom{n}{k} 2^{n-k}.$$

6. kMn

Let k, M, n be nonnegative integers such that $k + M \leq n$. Give a combinatorial proof that the equality

$$\binom{n}{M}\binom{n-M}{k} = \binom{n}{k}\binom{n-k}{M}$$

holds.