MATH3250 PRESENTATION PROBLEMS WEEK 2

Presentation Instruction. The following problems may be chosen for presentation problems during class. If you have attempted the problem before class (even if you don't have complete solution), you can volunteer to present the problems in a group of size 1 or 2. You can also volunteer to earn audience participation that day instead.

Reference the first chapter and second chapter of Bóna's "A Walk Through Combinatorics" textbook (4th, 3rd, or 2nd Ed.) for Pigeon-Hole Principle and induction methods.

Future homework and exam. A subset of these presentation problems will be chosen for future homework and exam problems.

1. Soccer Team

A soccer team scored a total of 40 goals this season. Nine players scored at least one of those goals. Prove that there are two players among those nine who scored the same number of goals.

Proof. □

2. Faculty members

A college has 39 departments, and a total of 262 faculty members in those departments. Prove that there are three departments in this college that have a total of at least 21 faculty members.

Proof. □

3. Five real numbers whose sum is 100

Let's say I give you a mystery set of five positive real numbers whose sum is 100. Prove that there are two numbers among them whose difference is at most 10.

Proof. □

4. Fridays

The month of January 2020 has five Fridays. (Optional: How many months in 2020 contain five Fridays?) For any given year, use the Pigeon-hole Principle to determine the possible number of months that contain five Fridays. See the source code for hints:

Proof. Insert proof □

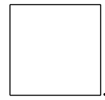
5. polygon

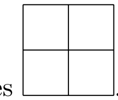
Prove (using induction) that the sum of the angles of a convex n-gon is $(n-2)180$ degrees.

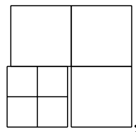
Note: You may use the fact that the sum of angles of a convex triangle is 180^o without proof.

Proof. Insert proof □

6. Squares

We start with one square piece of paper .

We then cut this square paper into four smaller squares .

Then cut one of the obtained small squares into four smaller squares, so that we get 7 squares , and so on.

Let a_n be the number of squares we have after the n-th time that we perform this cutting operation. Compute the first few values of a_n (beyond $n = 1$ and $n = 2$ given above), then conjecture an explicit formula for a_n, and then prove the formula using induction.

Proof. Insert proof □

7. Recurrence Relation

7.1. **Sample Solution.** Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and let

$$a_{n+1} = 2(a_0 + a_1 + \cdots + a_n)$$

for $n \geq 0$. Compute the first few values of a_n, then conjecture an explicit formula for a_n, and then prove the formula using induction.

Solution and proof. We claim that $a_n = 2 \cdot 3^{n-1}$ for $n \geq 1$. We prove this by strong induction on n. Since $2(a_0) = 2(1) = 2 \cdot 3^{1-1}$, the initial case (for $n = 1$) is verified. Now let us assume that the statement is true for all positive integers that are less than or equal to n. Then, we have

$$\begin{aligned}
a_{n+1} &= 2(a_0 + a_1 + a_2 + \cdots + a_n) \quad \text{by the recurrence relation} \\
&= 2a_0 + 2(a_1 + a_2 + \cdots + a_n) \\
&= 2 + 2(2 \cdot 1 + 2 \cdot 3 + \cdots + 2 \cdot 3^{n-1}) \quad \text{by the induction hypothesis} \\
&= 2 + 4(1 + 3 + \cdots + 3^{n-1}) \\
&= 2 + 4\left(\frac{3^n - 1}{2}\right) \quad \text{since the series is a geometric series} \\
&= 2 + 2(3^n - 1) \\
&= 2 \cdot 3^n.
\end{aligned}$$

This proves that our explicit formula is correct for $n + 1$, and the proof is complete. □

7.2. **A recurrence relation problem.** Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and

$$a_n = 3(a_0 + a_1 + \cdots + a_{n-1}) + 1$$

for $n > 0$. Compute the first few values of a_n, then conjecture an explicit formula for a_n, and then prove the formula using induction.

See hint in source code.

Proof. Insert proof □

8. Divisible by three

Use strong induction to prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Proof. Insert proof □